How to model Operational Risk?

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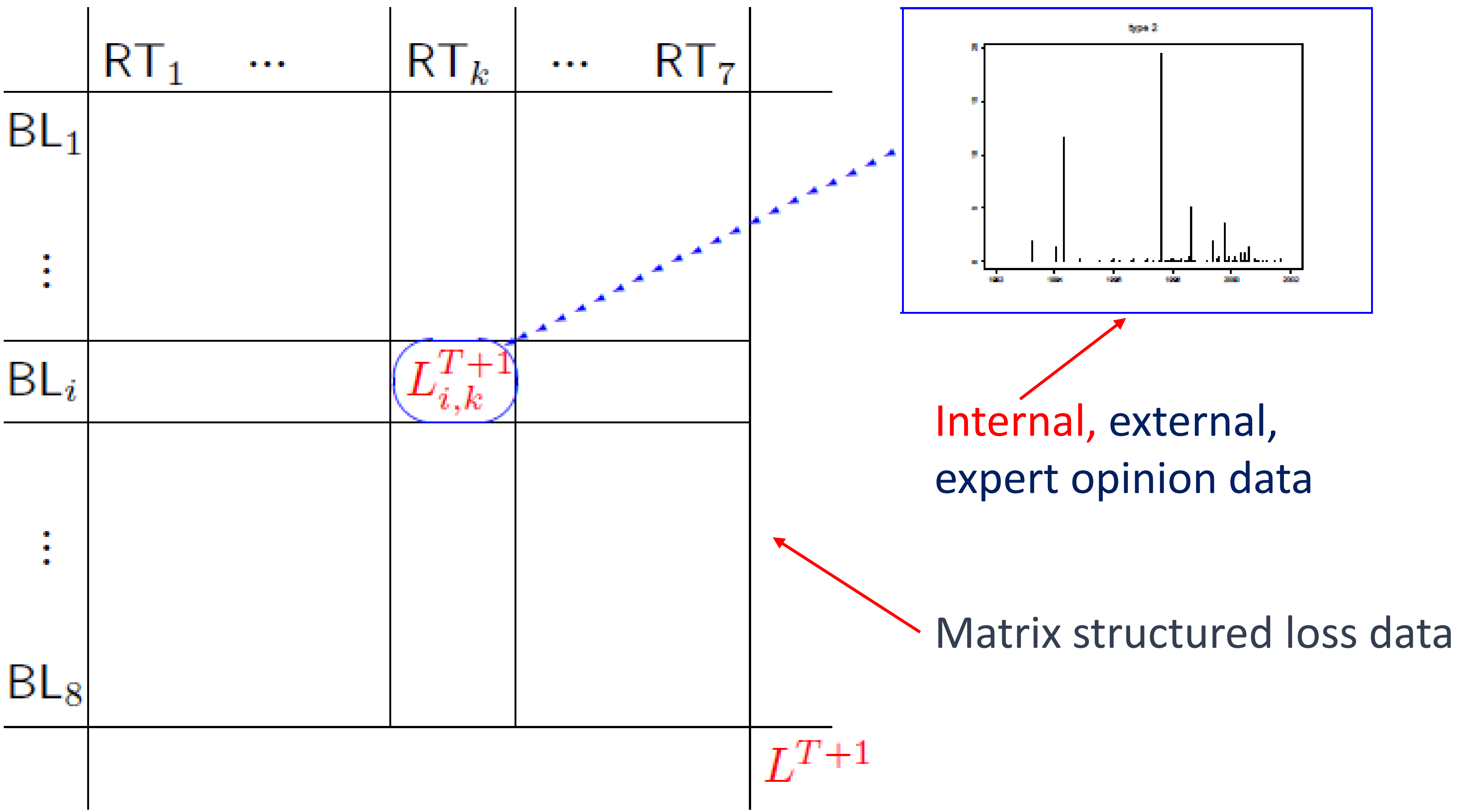
Risk Components (Basel II) now Basel III even Basel 3.5 …

- **Credit Risk**
- **Market Risk**
- **Operational Risk**
- **Business Risk** …

Operational Risk: The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. Including legal risk, but excluding strategic and reputational risk.

Loss Distribution Approach (LDA) within AMA-Framework



A complicated stochastic structure

"Insurance Analytics"

$$
\begin{cases}
L^{T+1} = \sum_{i=1}^{8} \sum_{k=1}^{7} L_{i,k}^{T+1} \\
L_{i,k}^{T+1} = \sum_{\ell=1}^{N_{i,k}^{T+1}} X_{i,k}^{\ell} \\
X_{i,k}^{\ell} : \text{loss severities} \\
N_{i,k}^{T+1} : \text{loss frequencies}
\end{cases}
$$

together with left-censoring, reporting delays (IBNR-like), insurance cover, ...

The two relevant (regulatory) risk measures:

Value-at-Risk (VaR) and Expected Shortfall (ES)

$\mathrm{VaR}_p(X)$

For $p \in (0,1)$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$$

$\mathrm{ES}_p(X)$

For $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F\text{ cont.})}{=} \mathbb{E}\left[X|X > \mathrm{VaR}_p(X)\right]$$

Basel II - Guidelines

- **Risk measure:** VaR (= Value-at-Risk, a quantile)
- **Time horizon:** 1 year
- **Level:** 99.9% (1 in 1000 year event!)

(Extreme event!)

"Darwinism"

- **Otherwise:** Full methodological freedom (within LDA)

How to model Operational Risk …

… if you must!

- Discussion between "Yes we can" and "No you can't"
- Banking versus Insurance:

  An example: Lausanne 2006 → BPV, EBK, FINMA …

- The record loss as of today: BoA's 16.65 billion USD settlement with the DOJ (August 2014), of which14.54 billion USD corresponds to BCBS Event type "Suitability, disclosure and fiduciary" and Business Line "Trading and sales"
- One thing is for sure:

  Operational Risk is of paramount importance!

  But how reliably can it be quantitatively risk managed?

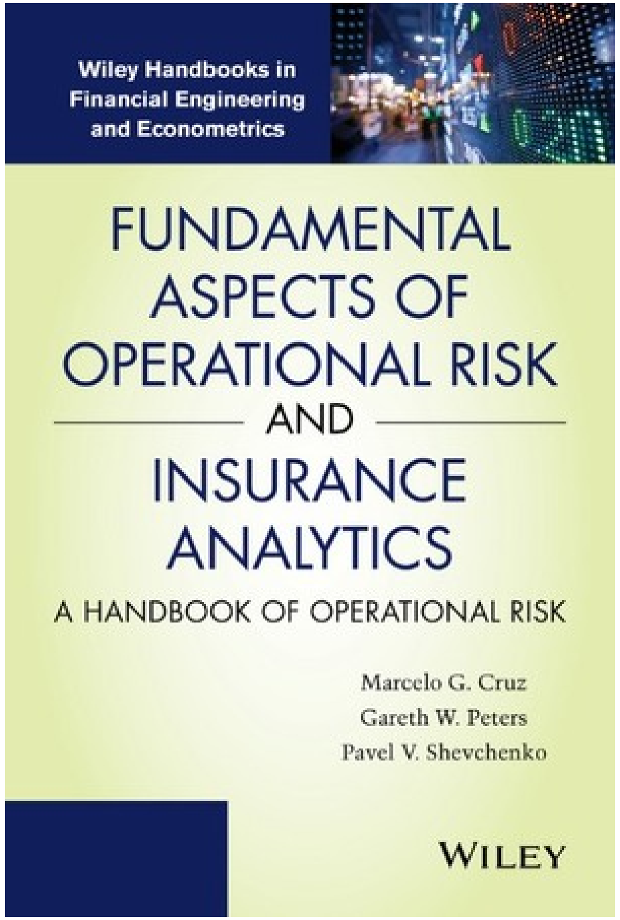
A quote from RISK.net, 13 March 2013:

- "In the past three years, we have seen, again and again, massive legal claims against banks that dwarf the sum of all the other operational risk loss events. That's a major issue, and I don't think many of the current risk models are reflecting this reality," says Paul Embrechts, professor of mathematics at ETH, a university in Zürich.
- He is referring to cases such as those arising from the pre-crisis mortgage boom, which produced a $25 billion settlement in February 2012 between the US and five mortgage servicers: Ally Financial, BAML, Citi, JP Morgan and Wells Fargo. More recent regulatory settlements include December's $1.9 billion money-laundering penalty for HSBC and the $1.5 billion Libor rigging fine for UBS. With US banks' mortgage misdeeds still not fully settled, and regulators around the world still pursuing Libor investigations – while civil cases wait in the wings – the pain is likely to continue.

Quotes from "Bank Capital for Operational Risk: A Tale of Fragility and Instability", M. Ames, T. Schuermann, H.S. Scott, February 10, 2014:

- On May 16, 2012, Thomas Curry, the Comptroller of the Currency (head of the OCC), said in a speech that bank supervisors are seeing "operational risk eclipse credit risk as a safety and soundness challenge." This represents a real departure from the past when concern was primarily focused on credit and market risk. A major component of operational risk is legal liability, and the recent financial crisis, a credit crisis par excellence, has been followed by wave after wave of legal settlements from incidents related to the crisis.
- To again quote Curry (2012), "The risk of operational failure is embedded in every activity and product of an institution."

As a consequence, a lot has been written on the topic:



2015, 900 pages!

etc …

The regulatory approaches towards OpRisk capital :

The Elementary Approaches:

- The Basic Indicator Approach $\mathrm{RC}^t_{\mathrm{BI}}(\mathrm{OR}) = \frac{1}{Z_t}\sum_{i=1}^{3} \alpha \max(\mathrm{GI}^{t-i}, 0)$

risk weight 15%

where $Z_t = \sum_{i=1}^{3} I_{\{\mathrm{GI}^{t-i}>0\}}$ and GI = Gross Income (year t-i)

- The Standardized Approach

$$\mathrm{RC}^t_{\mathrm{S}}(\mathrm{OR}) = \frac{1}{3}\sum_{i=1}^{3} \max\left[\sum_{j=1}^{8} \beta_j \mathrm{GI}_j^{t-i}, 0\right]$$

where the regulatory weight factors 12% ≤ β_j ≤ 18%, j = 1, … , 8 (BLs)
Note: recent BCBS document yields different weights and suggest replacing GI (Gross Income) by a new, so-called Business Indicator (BI).

The Advanced Approaches: AMA and in particular LDA → next slide

The Main LDA-Steps towards a Total Capital Charge

(LDA = Loss Distribution Approach, within AMA = Advanced Measurement Approach)

- Estimation of marginal VaR: $$\widehat{\mathrm{VaR}}_{\alpha}^{1}, \ldots, \widehat{\mathrm{VaR}}_{\alpha}^{d} \quad (1)$$

(α = p throughout)

- Additional Aggregation: $$\widehat{\mathrm{VaR}}_{\alpha}^{+} = \sum_{k=1}^{d} \widehat{\mathrm{VaR}}_{\alpha}^{k} \quad (2)$$

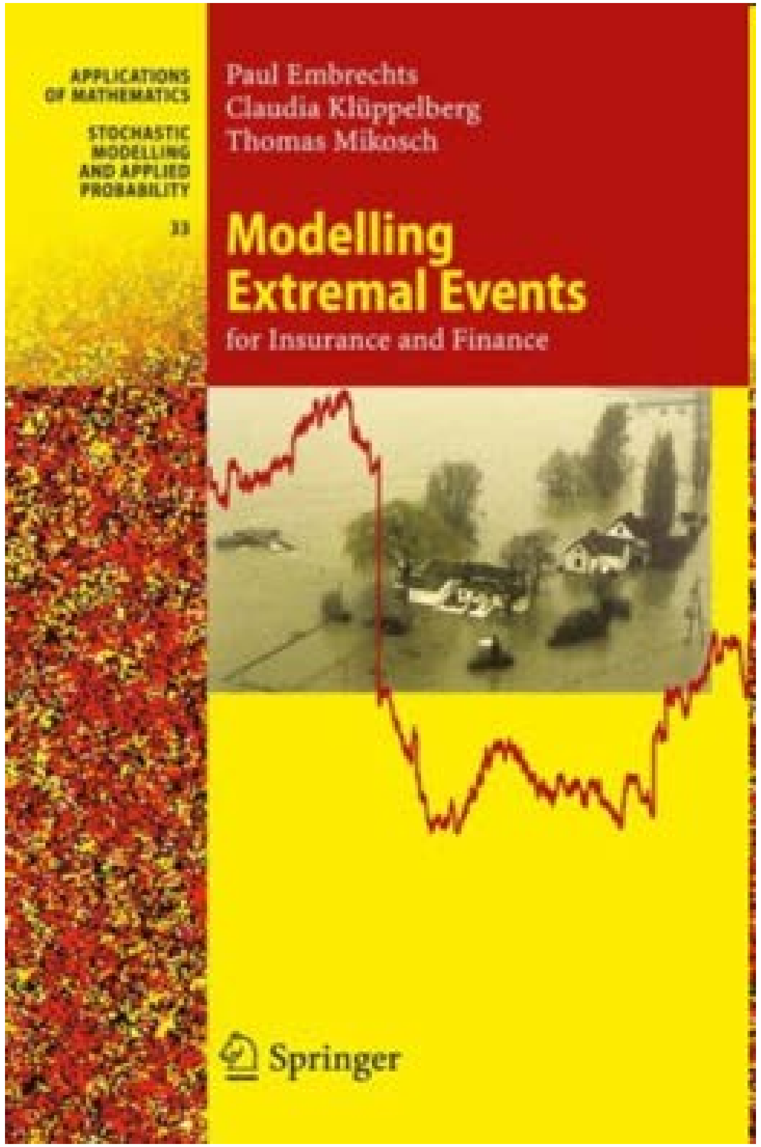
Two very big IFs

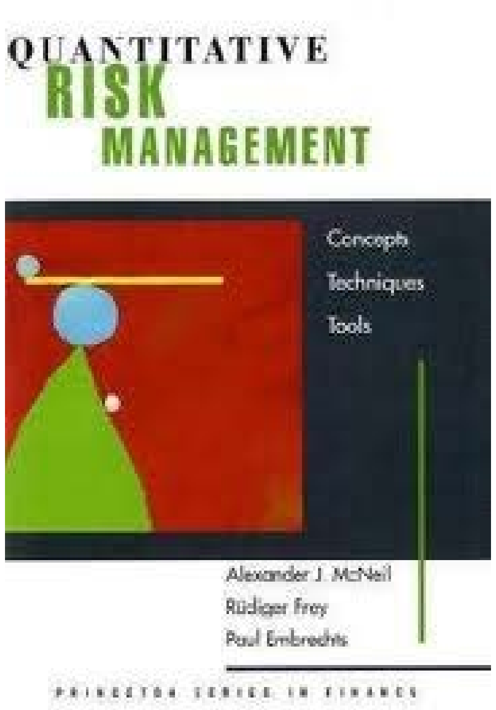
- Diversification: Operational Risk Capital = $$\mathrm{VaR}_{\alpha}^{\mathrm{real}} \overset{?}{<} \widehat{\mathrm{VaR}}_{\alpha}^{+} \quad (3)$$

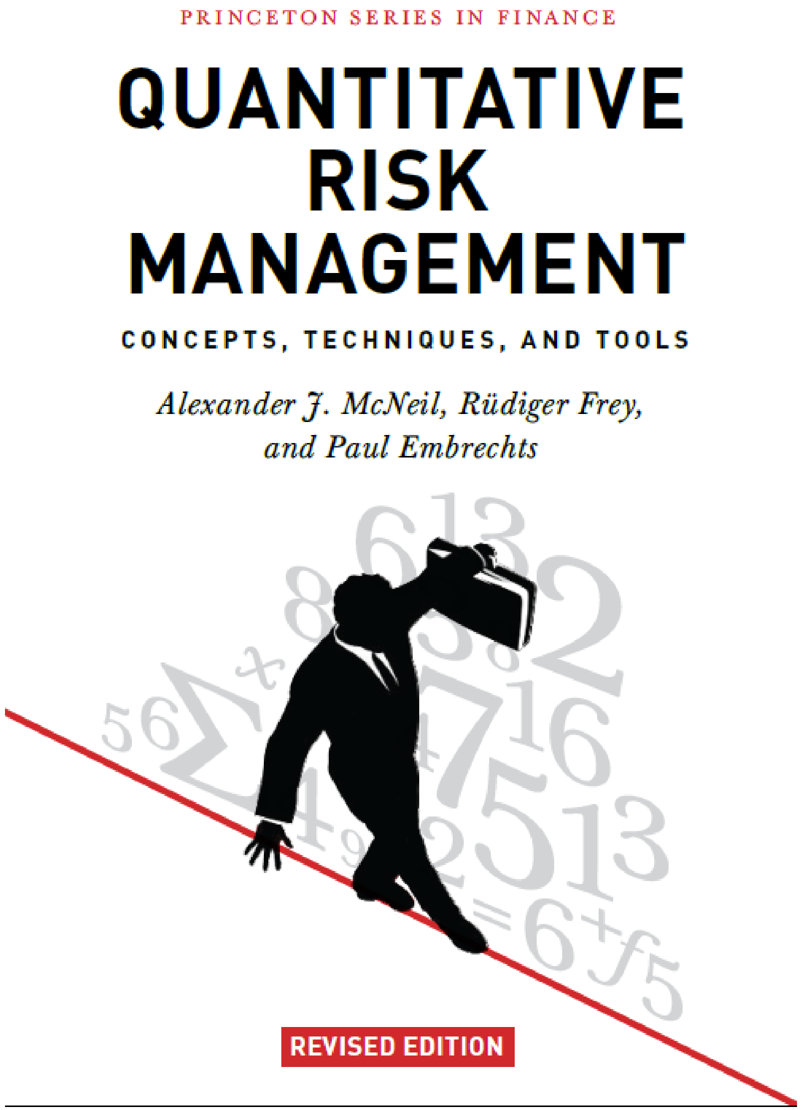
Some comments on (1), (2) and (3)

- For (1), estimating extreme quantiles, an EVT-based picture tells a thousand words → next two slides!
- Equation (2) is fully understood: Given that d risks are comonotone, then the VaR of their sum is the sum of their VaRs, hence (2) yields the VaR of the aggregate position under comonotonicity (“maximal correlation, perfect positive dependence, … ”)
- Definition: Random variables $X_1, \dots, X_d$ are comonotone if there exists a random variable Z and d increasing functions $\xi_1, \dots, \xi_d$ so that $X_i = \xi_i(Z)$, almost surely, i=1, … , d.
- For (3): model - and dependence uncertainty (← this talk)

(1) Estimating extreme quantiles (VaR)



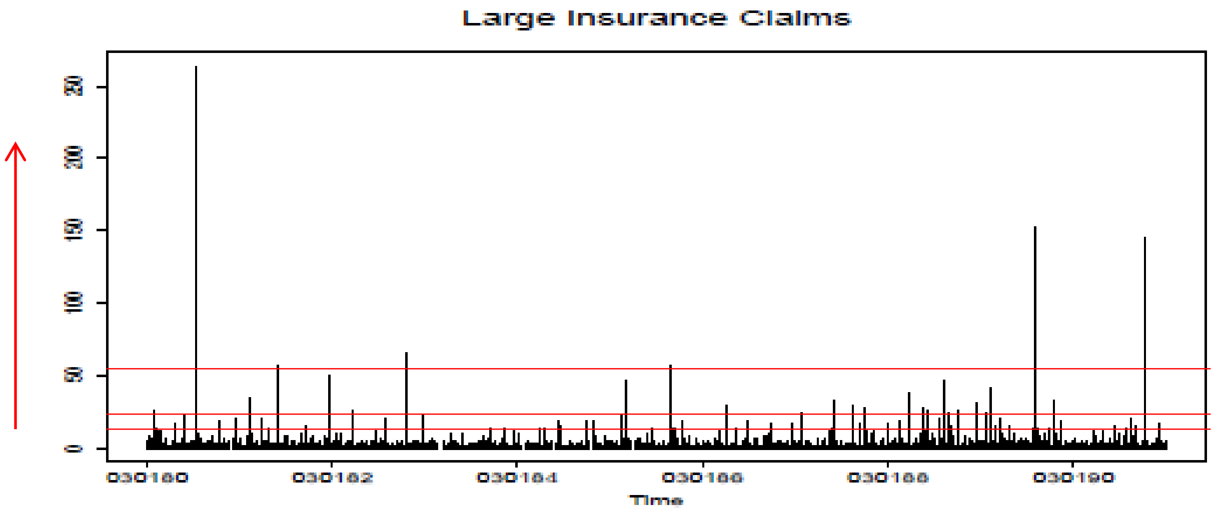




Danish Fire Loss Example

The Danish data consist of 2167 losses exceeding one million Danish Krone from the years 1980 to 1990. The loss figure is a total loss for the event concerned and includes damage to buildings, damage to contents of buildings as well as loss of profits. The data have been adjusted for inflation to reflect 1985 values.

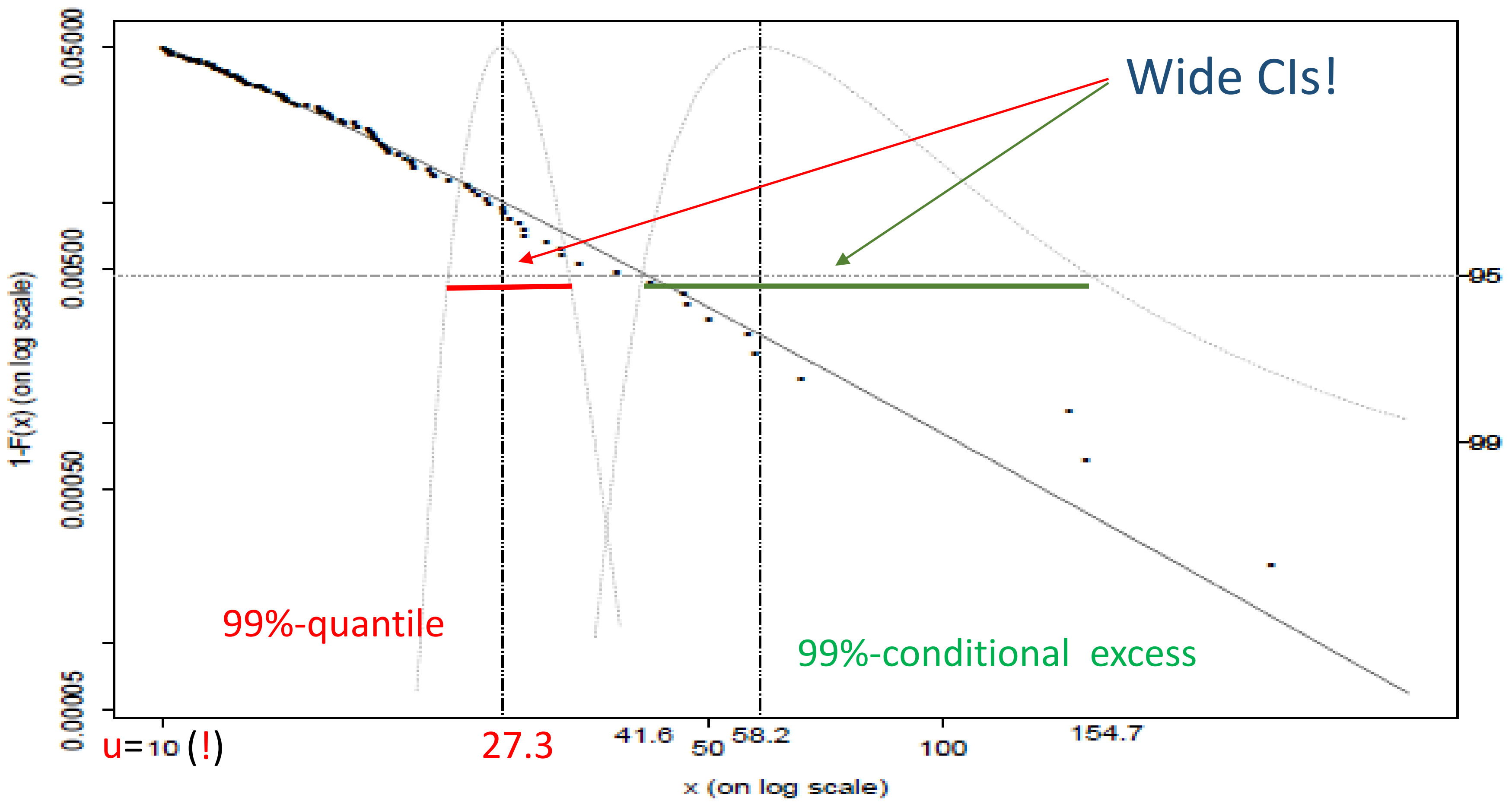
Very similar to OpRisk data!



POT u

99%-quantile with 95% CI (Profile Likelihood):
27.3 (23.3, 33.1)

99% Conditional Excess: E(X I X > 27.3) with CI



Concerning (2), recall that

- In general, VaR is not sub-additive, typical such cases occur for risks which are either very heavy-tailed (infinite mean), very skewed or (whatever the marginal dfs, e.g. N(0,1) or Exp(1)) have special dependence: all these cases are relevant for OpRisk!
- VaR is sub-additive (coherent) for multivariate elliptical risk factors.
- VaR and (hence also) ES are additive for comonotonic risks.
- Hence for ES, adding up the ES-contributions from the marginal risk factors always yields an upper bound for ES of the sum, and the upper bound is reached in the comonotonic case.
- For VaR this is NOT TRUE and this is relevant within the OpRisk context!

(3) Model - and Dependence Uncertainty

- Standard Basel II(+) procedure: aggregate the OpRisk losses BL-wise
- Estimate the resulting (8) VaRs
- Add these numbers up leading to a global estimate VaR^+
- Recall de notion and importance of comonotonic dependence
- Invoke a diversification argument to bring down regulatory capital from VaR^+ to a factor (1 – δ) VaR^+ where often δ ≈ 0.3
- However the non-convexity of VaR as a Risk measure may lead to true measures of risk (capital) larger than VaR^+, hence an important question concerns the problem of calculating best-worst bounds on risk measures of portfolio positions in general and VaR and ES more in particular

A fundamental problem in Quantitative Risk Management:

- Risk factors: $\mathbf{X} = (X_1, \ldots, X_d)$
- Model assumption: $X_i \sim F_i$, F_i known, $i = 1, \ldots, d$
- A financial position $\Psi(\mathbf{X})$
- A risk measure/pricing function: $\rho(\Psi(\mathbf{X}))$

Calculate $\rho(\Psi(\mathbf{X}))$

also denoted by S_d

Example:

- $\Psi(\mathbf{X}) = \sum_{i=1}^{d} X_i$
- $\rho = \mathrm{VaR}_p$ or $\rho = \mathrm{ES}_p$

Challenge:

- We need a *joint* model for the random vector $\mathbf{X}$
- Joint models are hard to get by

We will focus on the above special choices of Ψ and ρ.

For a given risk measure ρ denote

$$\overline{\rho}(S_d) = \sup \{ \rho(\textstyle\sum_{i=1}^{d} X_i): X_i \sim F_i, i = 1, \dots, d\}$$

and similarly

$$\underline{\rho}(S_d) = \inf \{ \rho(\textstyle\sum_{i=1}^{d} X_i): X_i \sim F_i, i = 1, \dots, d\}$$

where sup/inf are taken over all joint distribution models for the random vector $(X_1, \dots, X_d)$ with given marginal dfs $(F_1, \dots, F_d)$, or equivalently over all d-dimensional copulas.

We will consider as special cases the construction of the ranges:

$$(\underline{\mathrm{VaR}}, \overline{\mathrm{VaR}}) \text{ and } (\underline{\mathrm{ES}}, \overline{\mathrm{ES}})$$

known: comonotonic case

referred to as dependence-uncertainty ranges.

Summary of existing results:

$d = 2$:

- fully solved analytically

$d \geq 3$:

- Homogeneous model ($F_1 = \cdots = F_d$)
    - $\underline{\mathrm{ES}}_p(S_d)$ solved analytically for decreasing densities, e.g. Pareto, Exponential
    - $\overline{\mathrm{VaR}}_p(S_d)$ solved analytically for tail-decreasing densities, e.g. Pareto, Gamma, Log-normal
- Inhomogeneous model
    - Few analytical results: current research
- Numerical methods available: Rearrangement Algorithm

Sharp(!) bounds in the homogeneous case:

Sharp VaR bounds (Wang, Peng and Yang, 2013)

Suppose that the density function of F is decreasing on $[b, \infty)$ for some $b \in \mathbb{R}$. Then, for $p \in [F(b), 1)$, and $X \overset{d}{\sim} F$,

$$\overline{\mathrm{VaR}}_p(S_d) = d\mathbb{E}[X | X \in [F^{-1}(p + (d-1)c), F^{-1}(1-c)]],$$

where c is the smallest number in $[0, \frac{1}{d}(1-p)]$ such that

$$\int_{p+(d-1)c}^{1-c} F^{-1}(t)\mathrm{d}t \geq \frac{1-p-dc}{d}((d-1)F^{-1}(p+(d-1)c) + F^{-1}(1-c)).$$

Condition!

Red part clearly has an ES-type form.

- $c = 0$: $\overline{\mathrm{VaR}}_p(S_d) = \overline{\mathrm{ES}}_p(S_d)$.

More general result in the background!

Sharp VaR bounds II

Suppose that the density function of F is decreasing on its support. Then for $p \in (0,1)$ and $X \overset{d}{\sim} F$,

$$\underline{\mathrm{VaR}}_p(S_d) = \max\{(d-1)F^{-1}(0) + F^{-1}(p), d\mathbb{E}[X|X \le F^{-1}(p)]\}.$$

Sharp ES bounds (Bernard, Jiang and Wang, 2014)

Suppose that the density function of F is decreasing on its support. Then for $p \in (1-dc, 1)$, $q = (1-p)/d$ and $X \overset{d}{\sim} F$,

$$\underline{\mathrm{ES}}_p(S_d) = \frac{1}{q}\int_0^q \left((d-1)F^{-1}((d-1)t) + F^{-1}(1-t)\right) \mathrm{d}t,$$
$$= (d-1)^2 \mathrm{LES}_{(d-1)q}(X) + \mathrm{ES}_{1-q}(X),$$

where c is the smallest number in $[0, \frac{1}{d}]$ such that

$$\int_{(d-1)c}^{1-c} F^{-1}(t)\mathrm{d}t \ge \frac{1-dc}{d}((d-1)F^{-1}((d-1)c) + F^{-1}(1-c)).$$

- One large outcome is coupled with $d-1$ small outcomes. ⟵ : basic idea behind the proof

Stronger condition!

Left-tail-ES

Bounds in the inhomogeneous case: RA

Rearrangement Algorithm (RA): Embrecths, Puccetti and Rüschendorf (2013).

- A fast numerical procedure
- Based on the CM-idea
- Discretization of relevant quantile regions
- d possibly large
- Applicable to $\overline{\mathrm{VaR}}_p$, $\underline{\mathrm{VaR}}_p$ and $\underline{\mathrm{ES}}_p$

CM = Complete Mixability

Complete mixability, Wang and Wang (2011)

A distribution function F on $\mathbb{R}$ is called d-completely mixable (d-CM) if there exist d random variables $X_1, \ldots, X_d \sim F$ such that

$$\mathbb{P}(X_1 + \cdots + X_d = dk) = 1,$$

for some $k \in \mathbb{R}$.

Related concepts:

- d-mixability
- inhomogeneous case
- strong negative dependence
- general extremal dependence, …

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The Rearrangement Algorithm project

The Rearrangement Algorithm (RA) is an algorithm which has been introduced in [1] to compute numerically sharp lower and upper bounds on the distribution of a function of a number of dependent random variables having fixed marginal distributions.

The algorithm has been then developed further to:

- compute sharp bounds for the VaR/ES of high-dimensional portfolios having fixed marginal distributions; see [2], [3].
- compute sharp lower and upper bounds on the expected value of a supermodular function of d random variables having fixed marginal distributions; see [4].

For full details, see https://sites.google.com/site/rearrangementalgorithm/

Example 1: $P(X_i > x) = (1 + x)^{-2}$, $x \geq 0$, i = 1 , … , d

Bounds on VaR and ES for the sum of d Pareto(2) distributed rvs for $p = 0.999$; VaR_p^+ corresponds to the comonotonic case.

DU-gaps

434

220

can be explained

	$d = 8$	$d = 56$
$\underline{\mathrm{VaR}}_p$	31	53
$\underline{\mathrm{ES}}_p$	178	472
VaR_p^+	245	1715
$\overline{\mathrm{VaR}}_p$	465	3454
$\overline{\mathrm{ES}}_p$	498	3486
$\overline{\mathrm{VaR}}_p/\mathrm{VaR}_p^+$	1.898	2.014
$\overline{\mathrm{ES}}_p/\overline{\mathrm{VaR}}_p$	1.071	1.009

Comonotonic case: sum of marginal VaRs = d x marginal VaR

Comonotonic case: sum of marginal ESs = d x marginal ES

+/- factor 2 can be explained: Karamata's Theorem

+/- factor 1 can be explained : next slide

Two theorems (Embrechts, Wang, Wang, 2104):

Theorem 1: *Suppose the continuous distributions F_i, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $p \in (0,1)$,*

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ *is uniformly bounded for some* $k > 1$;

(ii) $\liminf_{d\to\infty} \frac{1}{d}\sum_{i=1}^{d} \mathrm{ES}_p(X_i) > 0.$

Then as $d \to \infty$,

$$\frac{\overline{\mathrm{ES}}_p(S_d)}{\overline{\mathrm{VaR}}_p(S_d)} = 1 + O(d^{1/k-1}).$$

Theorem 2: *Take* $1 > q \geq p > 0$. *Suppose that the continuous distributions* F_i, $i \in \mathbb{N}$, *satisfy (i) and (iii), and* $\limsup_{d\to\infty} \frac{\sum_{i=1}^{d} \mathbb{E}[X_i]}{\sum_{i=1}^{d} \mathrm{ES}_p(X_i)} < 1$, *then*

$$\liminf_{d\to\infty} \frac{\overline{\mathrm{VaR}}_q(S_d) - \underline{\mathrm{VaR}}_q(S_d)}{\overline{\mathrm{ES}}_p(S_d) - \underline{\mathrm{ES}}_p(S_d)} \geq 1.$$

A second example (inhomogeneous case):

ES and VaR of $S_d = X_1 + \cdots + X_d$, where

- $X_i \sim \text{Pareto}(2 + 0.1i)$, $i = 1, \ldots, 5$;
- $X_i \sim \text{Exp}(i - 5)$, $i = 6, \ldots, 10$;
- $X_i \sim \text{Log–Normal}(0, (0.1(i - 10))^2)$, $i = 11, \ldots, 20$.

	$d = 5$			$d = 20$		
	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\text{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\text{VaR}_{0.9875}$	12.06	56.21	44.16	22.12	126.63	104.51
$\text{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}}{\overline{\text{VaR}}_{0.975}}$		1.08			1.02	

Conclusions

- Operational Risk is a very important risk class, but defies reliable quantitative modelling
- More standardisation within the AMA/LDA is called for, do not allow for full modelling freedom: danger of backwards engineering
- Use lower confidence levels together with regulatory defined scaling
- Split legal risk from other Operational Risk classes and decide on separate treatment
- Make data available for scientific research
- Operational Risk type of data may lead to interesting statistical research questions which are relevant in a wider context, like (*) →

Some (extra) references:

- P. Embrechts, B. Wang, R. Wang (2014) Aggregation robustness and model uncertainty of regulatory risk measures. Finance and Stochastics, to appear (2015)
- (*) V. Chavez-Demoulin, P. Embrechts, M. Hofert (2014) An extreme value approach for modeling Operational Risk losses depending on covariates. Journal of Risk and Insurance, to appear (2015)

See www.math.ethz.ch/~embrechts for details.

Thank You!