Decision making in finance via stochastic dominance

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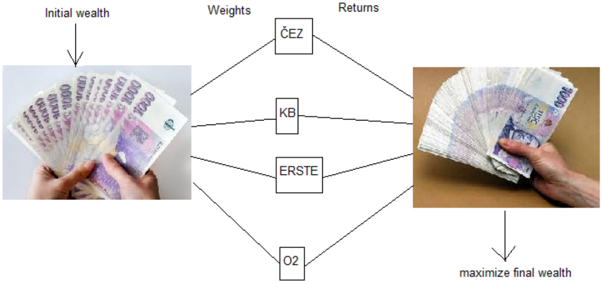
Motivation



Initial wealth
Weights
Returns
ČEZ
KB
ERSTE
O2
maximize final wealth

Motivation

- one criterion (expected wealth maximization) is not enough
- we do care about risk of the decision (investment)
- we want our decision to be better than some given decision (benchmark)
- the outcomes of decisions are random - **how to compare random variables**?
  - expected values - too weak
  - several characteristics - better, but still too weak
  - all realizations - too strong
  - compromise - STOCHASTIC DOMINANCE (SD)

Portfolio selection model

- Mean–risk models

$$\max_{\boldsymbol{\lambda}\in\Lambda} \quad m(\boldsymbol{\lambda}'\mathbf{r}) - \nu r(\boldsymbol{\lambda}'\mathbf{r})$$

or

$$\min_{\boldsymbol{\lambda}\in\Lambda} \quad r(\boldsymbol{\lambda}'\mathbf{r})$$
$$s.t. \quad m(\boldsymbol{\lambda}'\mathbf{r}) \geq \mu$$

- $\mathbf{r}$ is a random vector of assets returns
- maximizing mean $m(\boldsymbol{\lambda}'\mathbf{r})$ & minimizing risk $r(\boldsymbol{\lambda}'\mathbf{r})$
- risk measure - variance (semi variance,...,VaR, CVaR)
- risk or return parameter (ν, μ)

Portfolio selection model - alternative formulation

$$\max_{\boldsymbol{\lambda} \in \Lambda} \quad m(\boldsymbol{\lambda}'\mathbf{r})$$
$$s.t. \quad r(\boldsymbol{\lambda}'\mathbf{r}) \leq \rho$$

Disadvantage: the risk is controled by just one characteristic - not sufficient for non-gaussian distributions

Outline

We focus on:

- Nth order stochastic dominance relations, $N = 1, 2, 3$
- portfolio selection with Nth order stochastic dominance ($N = 1, 2, 3$) constraints
- empirical analysis for US stock market data (two periods - during crisis and after crisis)

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.97	1.10	1.41	1.16	0.18	1.11

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.97	1.10	1.41	1.16	0.18	1.11

Everybody? – Every non-satiated investor prefers B to A

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.97	1.04	1.41	1.14	0.19	1.56

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.97	1.04	1.41	1.14	0.19	1.56

Almost all of us? – Every non-satiated and risk averse investor prefers B to A.

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.95	1.04	1.40	1.13	0.24	1.46

Numerical example

	$S1$	$S2$	$S3$	μ	σ	sk
A	0.90	1.10	1.30	1.10	0.16	0.00
B	0.95	1.04	1.40	1.13	0.24	1.46

Most of us? – Every non-satiated, risk averse and skew-lover investor prefers B to A.

Notation

We consider a random vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)$ is the t-th row of matrix X representing the assets returns along t-th scenario. We assume that the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)^{\mathrm{T}}$ for a vector of portfolio weights and $X\boldsymbol{\lambda}$ represents returns of portfolio $\boldsymbol{\lambda}$. The portfolio possibilities are given by a simplex

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \lambda_j \geq 0, \ j = 1, 2, \ldots, N\}.$$

Benchmark

In all considered models, we compare the performance of a portfolio with the performance of a benchmark. In the simplest case, the comparison of mean returns (or risk measures) is considered. We consider: FSD, SSD and TSD relation for comparisons.

The benchmark portfolio is denoted by $\boldsymbol{\tau}$. It may be a current portfolio, a market portfolio (index), random goal,...

The feasible set consists of portfolios which outperformes the benchmark, no matter what kind of comparison is applied (FSD, SSD, TSD).

The objective is to maximize the mean return.

First order stochastic dominance (FSD) - notation

Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$.

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the first-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$) if

$$F_{\mathbf{r}'\boldsymbol{\lambda}}(x) \leq F_{\mathbf{r}'\boldsymbol{\tau}}(x) \qquad \forall x \in \mathbb{R}.$$

In general, FSD relation is expressed by infinitely many inequalities. However, under assumption of equiprobable scenarios, the number of inequalities is equal to the number of scenarios.

First order stochastic dominance (FSD) - interpretation

Other equivalent definitions: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all utility functions.
- No non-satiable decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F^{-1}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \geq F^{-1}_{\mathbf{r}'\boldsymbol{\tau}}(y) \quad \forall y \in [0, 1]$.
- $\mathrm{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\lambda}) \leq \mathrm{VaR}_{\alpha}(-\mathbf{r}'\boldsymbol{\tau}) \quad \forall \alpha \in [0, 1]$.

First order stochastic dominance (FSD) - discrete distribution

- Let X be a matrix of scenarios of asset returns. Then $X\boldsymbol{\lambda}$ are returns of portfolio $\boldsymbol{\lambda}$ and $X\boldsymbol{\tau}$ of portfolio $\boldsymbol{\tau}$
- Let $a_1 \leq a_2 \leq ... \leq a_N$ be the returns of portfolio $\boldsymbol{\lambda}$ and $b_1 \leq b_2 \leq ... \leq b_N$ be the returns of portfolio $\boldsymbol{\tau}$. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_i \geq b_i$, $i = 1, ..., N$.
- equivalently $X\boldsymbol{\lambda} \geq PX\boldsymbol{\tau}$ for at least one permutation matrix P, that is, binary matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

First order stochastic dominance (FSD) - continuous distributions

- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a gaussian (normal) distribution $N(\mu_{\boldsymbol{\lambda}}, \sigma_{\boldsymbol{\lambda}})$, $N(\mu_{\boldsymbol{\tau}}, \sigma_{\boldsymbol{\tau}})$, respecively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\mu_{\boldsymbol{\lambda}} \geq \mu_{\boldsymbol{\tau}}$ and $\sigma_{\boldsymbol{\lambda}} = \sigma_{\boldsymbol{\tau}}$
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a uniform distribution on interval $\langle a_{\boldsymbol{\lambda}}, b_{\boldsymbol{\lambda}} \rangle$, $\langle a_{\boldsymbol{\tau}}, b_{\boldsymbol{\tau}} \rangle$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_{\boldsymbol{\lambda}} \geq a_{\boldsymbol{\tau}}$ and $b_{\boldsymbol{\lambda}} \geq b_{\boldsymbol{\tau}}$.
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have an exponential distribution with mean value $m_{\boldsymbol{\lambda}}$, $m_{\boldsymbol{\tau}}$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $m_{\boldsymbol{\lambda}} \geq m_{\boldsymbol{\tau}}$.

Second order stochastic dominance – definitions

Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The twice cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) = \int_{-\infty}^{y} F_{\mathbf{r}'\boldsymbol{\lambda}}(x)dx. \qquad (1)$$

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the second-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \leq F^{(2)}_{\mathbf{r}'\boldsymbol{\tau}}(y) \qquad \forall y \in \mathbb{R}.$$

In general, also SSD relation is expressed by infinitely many inequalities. However, under assumption of equiprobable scenarios, the number of inequalities is equal to the number of scenarios.

Second order stochastic dominance – interpretation

Other equivalent definitions of SSD relation: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all concave utility functions.
- No non-satiable and risk averse decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F^{-2}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \geq F^{-2}_{\mathbf{r}'\boldsymbol{\tau}}(y) \;\; \forall y \in [0,1]$, where $F^{-2}_{\mathbf{r}'\boldsymbol{\lambda}}$ is a cumulated quantile function.
- $\mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) \leq \mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) \;\; \forall \alpha \in [0,1]$, where

$$\begin{aligned} \mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) = \min_{v \in \mathbb{R}, z_t \in \mathbb{R}^+} \quad & v + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T} z_t \\ \text{s.t.} \quad & z_t \geq -\mathbf{x}^t\boldsymbol{\lambda} - v, \quad t = 1,2,...,T \end{aligned}$$

Second order stochastic dominance (SSD) - discrete distribution

- Let X be a matrix of scenarios of asset returns. Then $X\boldsymbol{\lambda}$ are returns of portfolio $\boldsymbol{\lambda}$ and $X\boldsymbol{\tau}$ of portfolio $\boldsymbol{\tau}$
- Let $a_1 \leq a_2 \leq ... \leq a_N$ be the returns of portfolio $\boldsymbol{\lambda}$ and $b_1 \leq b_2 \leq ... \leq b_N$ be the returns of portfolio $\boldsymbol{\tau}$. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\sum_{j=1}^{i} a_j \geq \sum_{j=1}^{i} b_j$, $i = 1, ..., N$.
- equivalently $X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau}$ for at least one double stochastic matrix W, that is, non-negative matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

Second order stochastic dominance (SSD) - continuous distributions

- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a gaussian (normal) distribution $N(\mu_{\boldsymbol{\lambda}}, \sigma_{\boldsymbol{\lambda}})$, $N(\mu_{\boldsymbol{\tau}}, \sigma_{\boldsymbol{\tau}})$, respecively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\mu_{\boldsymbol{\lambda}} \geq \mu_{\boldsymbol{\tau}}$ and $\sigma_{\boldsymbol{\lambda}} \leq \sigma_{\boldsymbol{\tau}}$
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a uniform distribution on interval $\langle a_{\boldsymbol{\lambda}}, b_{\boldsymbol{\lambda}} \rangle$, $\langle a_{\boldsymbol{\tau}}, b_{\boldsymbol{\tau}} \rangle$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_{\boldsymbol{\lambda}} \geq a_{\boldsymbol{\tau}}$ and $a_{\boldsymbol{\lambda}} - a_{\boldsymbol{\tau}} \geq -b_{\boldsymbol{\lambda}} + b_{\boldsymbol{\tau}}$.
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have an exponential distribution with mean value $m_{\boldsymbol{\lambda}}$, $m_{\boldsymbol{\tau}}$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $m_{\boldsymbol{\lambda}} \geq m_{\boldsymbol{\tau}}$.

The three times cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as

$$F^{(3)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) = \int_{-\infty}^{y} F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(x)dx. \quad (2)$$

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the third-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{TSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if

$$F^{(3)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \leq F^{(3)}_{\mathbf{r}'\boldsymbol{\tau}}(y) \qquad \forall y \in \mathbb{R}$$

and $E\mathbf{r}'\boldsymbol{\lambda} \geq E\mathbf{r}'\boldsymbol{\tau}$.

In general, also TSD relation is expressed by infinitely many inequalities. Unlike the SSD or FSD relation, TSD relation can not be expressed by finately many inequalities in the case of a discrete distribution. Therefore Post and Kopa (2016) suggest a tight SCTSD approximation for $F^{(3)}_{\mathbf{r}'\boldsymbol{\lambda}}(y)$ using the fact that $F^{(3)}_{\mathbf{r}'\boldsymbol{\lambda}}(y)$ is increasing, convex, piecewise quadratic function (under assuption of a discrete distribution).

Third order stochastic dominance – interpretation

Other equivalent definitions of TSD relation: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{TSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all concave utility functions having $u'''(x) \geq 0$.
- $S_{\mathbf{r}'\boldsymbol{\lambda}}(x) \leq S_{\mathbf{r}'\boldsymbol{\tau}}(x)\ \forall y \in \mathbb{R}$ where S is a semivariance function, i.e. $S_{\mathbf{r}'\boldsymbol{\lambda}}(x) = E(\max(0, x - \mathbf{r}'\boldsymbol{\lambda}))^2$
- $S_{\mathbf{r}'\boldsymbol{\lambda}}(x) = 2F^{(3)}_{\mathbf{r}'\boldsymbol{\lambda}}(x)$

Portfolio selection models with SD constraints

$$\begin{aligned} &\max \mathbb{E}(\mathrm{r}'\boldsymbol{\lambda}) \\ &\text{s.t. } \mathrm{r}'\boldsymbol{\lambda} \succ_{FSD} \mathrm{r}'\boldsymbol{\tau} \\ &\boldsymbol{\lambda} \in \Lambda \end{aligned} \tag{3}$$

$$\begin{aligned} &\max \mathbb{E}(\mathrm{r}'\boldsymbol{\lambda}) \\ &\text{s.t. } \mathrm{r}'\boldsymbol{\lambda} \succ_{SSD} \mathrm{r}'\boldsymbol{\tau} \\ &\boldsymbol{\lambda} \in \Lambda \end{aligned} \tag{4}$$

$$\begin{aligned} &\max \mathbb{E}(\mathrm{r}'\boldsymbol{\lambda}) \\ &\text{s.t. } \mathrm{r}'\boldsymbol{\lambda} \succ_{TSD} \mathrm{r}'\boldsymbol{\tau} \\ &\boldsymbol{\lambda} \in \Lambda \end{aligned} \tag{5}$$

where $\boldsymbol{\tau}$ is a given benchmark (reference) portfolio. Since we consider a discrete distribution of returns with equiprobable scenarios we can follow Kuosmanen (2004) and reformulate (3) as a linear mixed integer programming problem:

Reformulation with FSD constraints

Following Kuosmanen (2004):

$$\max_{p_{i,j},\boldsymbol{\lambda}} \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}^t\boldsymbol{\lambda} \tag{6}$$

s.t.

$$X\boldsymbol{\lambda} \geq PX\boldsymbol{\tau}$$

$$P = \{p_{i,j}\}_{i,j=1}^{T}, \sum_{i=1}^{T} p_{i,j} = \sum_{j=1}^{T} p_{i,j} = 1$$

$$p_{i,j} \in \{0,1\}, \; i,j = 1,\ldots,T$$

$$\boldsymbol{\lambda} \in \Lambda.$$

If the probability scenarios would not be the same for all scenarios, see Luedtke (2008).

Reformulation with SSD constraints

Following Kopa and Post (2015):

$$\max_{d_s, v_{t,s}, \boldsymbol{\lambda}} \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}^t \boldsymbol{\lambda} \tag{7}$$

$$\text{s.t.} \quad -T^{-1}\mathbf{x}^t\boldsymbol{\lambda} + \frac{1}{s} d_s - v_{t,s} + \frac{1}{s} \sum_{k=1}^{T} v_{k,s} \leq -\frac{1}{Ts} \sum_{k=1}^{s} \mathbf{x}^k \boldsymbol{\tau}, \quad t, s = 1, 2, \ldots, T$$

$$d_s, \ v_{t,s} \geq 0, t, s = 1, 2, \ldots, T$$

$$\boldsymbol{\lambda} \in \Lambda$$

If the probability scenarios would not be the same for all scenarios, see Dentcheva and Ruszczynski (2003, 2006) or Luedtke (2008).

Reformulation with TSD constraints

Following Post and Kopa (2017):

$$\max_{\theta_{k,t},\boldsymbol{\lambda}} \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^t\boldsymbol{\lambda} \tag{8}$$

$$\text{s.t.} \quad \frac{1+e_k}{T}\sum_{t=1}^{T}\theta_{k,t}^2 \leq 2F^{(3)}_{\mathbf{r}'\boldsymbol{\tau}}(z_k), \qquad k=1,\dots,K$$

$$\theta_{k,t} \geq z_k - \mathbf{x}^t\boldsymbol{\lambda} \qquad k=1,\dots,K,\ t=1,\dots,T$$

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^t\boldsymbol{\lambda} \geq \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^t\boldsymbol{\tau}$$

$$\boldsymbol{\lambda} \in \Lambda$$

$$\theta_{k,t} \geq 0, \qquad k=1,\dots,K,\ t=1,\dots,T$$

where $e_1 = e_2 = 0$ and

$$e_k = \frac{F^{(3)}_{\mathbf{r}'\boldsymbol{\tau}}(z_k)}{F^{(3)}_{\mathbf{r}'\boldsymbol{\tau}}(z_{k-1}) + F^{(2)}_{\mathbf{r}'\boldsymbol{\tau}}(z_{k-1})(z_k - z_{k-1})} - 1, \qquad k=3,\dots,K.$$

Numerical study - description

- In this section we apply problems (6)-(8) to monthly returns of 25 Fama-French portfolios (base assets) formed on size and book-to-market, such that the portfolios are the intersections of 5 portfolios formed on size (market equity, ME) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME).
- Moreover we include returns of one-year US tbill as the risk free returns.
- The reference (benchmark) portfolio τ is the market US portfolio, proxied by the CRSP index.
- We take two data sets: one from the during crisis period 2007-2010 and the other one from the after crisis period 2011-2014 (each period consists of 48 scenarios of returns).

Numerical study - results

Table 1: Risk-reward profile of solution portfolios - during crisis period

Portfolio	Mean	Standard deviation
Benchmark	0.1846	5.7801
FSD solution portfolio	0.1846	5.7801
SSD solution portfolio	0.8963	6.1931
TSD solution portfolio	0.9216	6.4098

Table 2: Risk-reward profile of solution portfolios - after crisis period

Portfolio	Mean	Standard deviation
Benchmark	1.2463	3.4017
FSD solution portfolio	1.2463	3.4017
SSD solution portfolio	1.3333	3.3271
TSD solution portfolio	1.3501	3.5879

Endogenous randomness

- Uncertainty in stochastic optimization models:
  - endogenous
  - exogenous.
- Endogenous uncertainty is inner uncertainty of the model.
- the random (uncertain) element of the problem may depend on the solution (decision).
- For example, the decision-maker can force one possibility to become more probable.
- Observed at illiquid markets, in high-frequency trading, deposit interest rates - Czech banks

Stochastic dominance with endogenous randomness

If we allow that ϱ may depend on the decision vector $\boldsymbol{\lambda}$:

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the second-order stochastic dominance with endogenous randomness ($\varrho(\boldsymbol{\lambda})'\boldsymbol{\lambda} \succ_{SSD} \varrho(\boldsymbol{\tau})'\boldsymbol{\tau}$) if

$$F^{(2)}_{\varrho(\boldsymbol{\lambda})'\boldsymbol{\lambda}}(t) \leq F^{(2)}_{\varrho(\boldsymbol{\tau})'\boldsymbol{\tau}}(t) \qquad \forall t \in \mathbb{R}.$$

Stochastic dominance with endogenous randomness

If endogenous randomness affects only the values of the return scenario matrix X but not the probabilities of the scenarios:

$$\varrho(\boldsymbol{\lambda})'\boldsymbol{\lambda} \succ_{SSD} \varrho(\boldsymbol{\tau})'\boldsymbol{\tau} \tag{1}$$

$$\Longleftrightarrow$$

$$\begin{aligned} \exists P = \{p\}_{i,j=1}^{T} : X(\boldsymbol{\lambda})\boldsymbol{\lambda} &\geq PX(\boldsymbol{\tau})\boldsymbol{\tau} \\ \sum_{i=1}^{T} p_{i,j} &= 1, j = 1, 2, ..., T \\ \sum_{j=1}^{T} p_{i,j} &= 1, i = 1, 2, ..., T \\ p_{i,j} &\geq 0. \end{aligned}$$

where $X(\boldsymbol{\lambda})$ is the scenario return matrix if portfolio $\boldsymbol{\lambda}$ is chosen. Similarly, if we invest in portfolio $\boldsymbol{\tau}$ the scenario return matrix is $X(\boldsymbol{\tau})$.

Portfolio selection problems - exogenous randomness

If exogenous randomness is assumed, the model takes the form:

$$\begin{aligned} & \max \mathbb{E}(\boldsymbol{\varrho}'\boldsymbol{\lambda}) && (2)\\ \text{s.t.} \quad & \boldsymbol{\varrho}'\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\varrho}'\boldsymbol{\tau} \\ & \boldsymbol{\lambda} \in \Lambda \end{aligned}$$

where $\boldsymbol{\tau}$ is a given benchmark (reference) portfolio.

If we consider a discrete distribution of returns with equiprobable scenarios (atoms):

$$\begin{aligned} & \max \mathbb{E}(\boldsymbol{\varrho}'\boldsymbol{\lambda}) && (3)\\ \text{s.t.} \quad & X\boldsymbol{\lambda} \geq PX\boldsymbol{\tau}, && (4)\\ & \sum_{i=1}^{T} p_{i,j} = 1, j = 1, 2, ..., T, && (5)\\ & \sum_{j=1}^{T} p_{i,j} = 1, i = 1, 2, ..., T, && (6)\\ & p_{i,j} \geq 0 && (7) \end{aligned}$$

Portfolio selection problems - endogenous randomness

In the case, that endogenous randomness affects only the values of the return scenario matrix X but not the probabilities of the scenarios:

$$\max \mathbb{E}(\varrho(\boldsymbol{\lambda})'\boldsymbol{\lambda}) \tag{9}$$

$$\text{s.t.} \quad X(\boldsymbol{\lambda})\boldsymbol{\lambda} \geq PX(\boldsymbol{\tau})\boldsymbol{\tau}, \tag{10}$$

$$\sum_{i=1}^{T} p_{i,j} = 1, j = 1, 2, ..., T, \tag{11}$$

$$\sum_{j=1}^{T} p_{i,j} = 1, i = 1, 2, ..., T, \tag{12}$$

$$p_{i,j} \geq 0 \tag{13}$$

$$\boldsymbol{\lambda} \in \Lambda \tag{14}$$

Numerical example

We assume only three assets with three equiprobable scenarios of returns (in percentage). In particular

$$X = \begin{pmatrix} 3 & 8 & 5 \\ 2 & 4 & 3 \\ 11 & 3 & 6 \end{pmatrix}$$

and the benchmark portfolio is $\boldsymbol{\tau} = (0, 0, 1)$.

Solution for exogenous randomness

Let's solve first the problem, that is, the model with exogenous randomness:

- In this case, it is easy to see that the first asset is the most profitable, so we would like to invest in it as much as possible.
- Since the minimal return of the benchmark portfolio is 3% the maximal possible (and optimal) weight of the first asset is 0.5 when combined with the second asset.
- Hence, the optimal portfolio $\boldsymbol{\lambda}^* = (0.5, 0.5, 0)$ and the optimal double stochastic matrix is the identity matrix.

Solution for endogenous randomness

Now assume that if a massive investment is done in the first (the second, the third) asset, that is $\lambda_1 \geq 0.9$ ($\lambda_2 \geq 0.9$, $\lambda_3 \geq 0.9$) the return of the first (the second, the third) asset is increased by 1 for the last scenario. Consider the following subsets of Λ:

$$\Lambda_1 = \{\boldsymbol{\lambda} \in \Lambda : \lambda_1 \geq 0.9\}, \;\; \Lambda_2 = \{\boldsymbol{\lambda} \in \Lambda : \lambda_2 \geq 0.9\}, \;\; \Lambda_3 = \{\boldsymbol{\lambda} \in \Lambda : \lambda_3 \geq 0.9\}.$$

Now we can express the decision dependent random returns:

$$X(\boldsymbol{\lambda}) = \begin{pmatrix} 3 & 8 & 5 \\ 2 & 4 & 3 \\ 11 & 3 & 6 \end{pmatrix} + \mathcal{I}(\boldsymbol{\lambda} \in \Lambda_1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathcal{I}(\boldsymbol{\lambda} \in \Lambda_2) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+ \mathcal{I}(\boldsymbol{\lambda} \in \Lambda_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution for endogenous randomness

The problem can be decomposed in four separate problems which are very similar to each other, just the constraint $\boldsymbol{\lambda} \in \Lambda$ is modified as follows:

1. $\boldsymbol{\lambda} \in \Lambda_1$
2. $\boldsymbol{\lambda} \in \Lambda_2$
3. $\boldsymbol{\lambda} \in \Lambda_3$
4. $\boldsymbol{\lambda} \in \Lambda \backslash (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$.

- When it is replaced by $\boldsymbol{\lambda} \in \Lambda_1$, no feasible solution exists, because the smallest return of any portfolio $\boldsymbol{\lambda} \in \Lambda_1$ is smaller than the smallest return of the benchmark.
- If $\boldsymbol{\lambda} \in \Lambda_2$ then the optimal portfolios are $\boldsymbol{\lambda}^2 = (1-k, k, 0)$, $k \in [0.9, 1]$ and the optimal objective value is $16/3$.
- If $\boldsymbol{\lambda} \in \Lambda_3$) then the optimal solution is $\boldsymbol{\lambda}^3 = (0.05, 0.05, 0.9)$ with optimal objective value smaller than $16/3$.
- Finally, $\boldsymbol{\lambda} \in \Lambda \backslash (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$ gives the same optimal solution as the exogenous randomness case: $\boldsymbol{\lambda}^4 = (0.5, 0.5, 0)$ with optimal objective value is $31/6$.
- Since $16/3 > 31/6$ the optimal solutions of the endogenous randomness portfolio selection model are $\boldsymbol{\lambda}^* = (1-k, k, 0)$, $k \in [0.9, 1]$.

Main related recent references

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