REINVENTING "GUTTMAN SCALING" AS A STATISTICAL MODEL: ABSOLUTE SIMPLEX THEORY

Peter M. Bentler*

University of California, Los Angeles

*With thanks to Prof. Ke-Hai Yuan, University of Notre Dame

Topics

(Not Precisely Ordered)

- Guttman Scales
- Absolute Simplex Theory (AST)
- A Unique Feature of Total Scores
- Mean Parameterization for AST
- Coefficient of Variation Parameterization for AST
- Moment Matrix based Estimation & Testing
- Unidimensionality
- Interval Scale
- Trait CDF
- Variety of AST Regression Estimation Methods

Guttman (1944) Scale

00000
00001
00011
00011
00011
00111
00111
00111
01111
11111

- Example of 10 subjects, 5 items
- “1” means correct (keyed) response
- Items ordered from hard to easy

Key point: If person gets a “hard” item right, he/she gets all easier items right

Largely abandoned – no clear statistical estimation and testing machinery

Early References

Walker, D. A. (1931). Answer-pattern and score-scatter in tests and examinations. *British Journal of Psychology*, *22*, 73-86. ("unig" answer patterns)

Guttman, L. (1944). A basis for scaling qualitative data. *American Sociological Review*, *9*, 139-150. ("scale" – now a "Guttman scale")

Loevinger, J. (1948). The technic of homogeneous tests compared with some aspects of "scale analysis" and factor analysis. *Psychological Bulletin*, *45*(6), 507-529. ("cumulative homogeneous scale")

Example: Self-reported Height

Answer "yes" or "no." Are you

(1) over 5' tall (152.4 cm); (2) over 5'2" tall (157.4 cm); (3) over 5'6" tall (168 cm), (4) over 6' tall (182.9 cm)?

With 1 = yes and 0 = no; and h_k is the k-th person's height, the only logically possible response patterns are:

0000	if $h_k \leq 5'$	($h_k \leq 152.4$ cm)
1000	if $5' < h_k \leq 5'2''$	($152.4 < h_k \leq 157.4$ cm)
1100	if $5'2'' < h_k \leq 5'6''$	($157.4 < h_k \leq 168$ cm)
1110	if $5'6'' < h_k \leq 6'$	($168 < h_k \leq 182.9$ cm)
1111	if $6' < h_k$	(182.9 cm $< h_k$)

Every other pattern makes no sense – it's an "error".

How about 1101? Then $h_k \leq 168$ cm, but also $h_k \geq 182.9$ cm

Absolute Simplex Theory (AST)
(Bentler, 1971)

- An *absolute simplex* is an n by p data matrix (n>p) that can be generated completely from one parameter per item. The parameters are "absolute" (cannot be rescaled without information loss)
- It is a parameterization, estimation, and testing machinery for Guttman and near-Guttman data
- Approach discussed today is based on recent developments, including structural equation modeling and regression (Bentler, 2009, 2011ab; Bentler & Yuan, 2011)

References

Bentler, P. M. (1971). An implicit metric for ordinal scales: Implications for assessment of cognitive growth. In D. R. Green, M. P. Ford, & G. B. Flamer (Eds.), *Measurement and Piaget* (pp. 34-63). New York: McGraw Hill.

Bentler, P. M. (2009). Estimation, tests, and extensions of a 1-parameter item scaling model. Paper presented at Society of Multivariate Experimental Psychology, Salishan, OR

Bentler, P. M. (2011a). Can Interval-level Scores be obtained from Binary Responses? Invited presentation at Western Psychological Association, LA. UCLA Statistics Preprint #622, http://preprints.stat.ucla.edu/

Bentler, P. M. (2011b). *SEM and regression estimation in the absolute simplex (Bentler-Guttman scale)*. Paper presented at International Meeting of the Psychometric Society, Hong Kong.

Bentler, P. M., & Yuan, K. –H. (2011). *Absolute simplex theory for unidimensional scaling of binary items*. Manuscript in preparation.

A Unique Feature of AST Total Scores

Let *X* be a *p*-vector with $E(X) = \mu$ and $Cov(X) = \Sigma$
A total score is $X_T = w'X$ where *w* is any fixed weights.

The ordering of persons by a total score X_T *in an AST population is invariant to choice of item weights* $w_i > 0$.
If the person ordering depends on item weights, the items do not arise from a strict AST population. Also,
$X_T \Rightarrow X$
There is no further information in the pattern of 0-1 responses.

To prove this, consider the Height data with weighted sum $X_T = w'X$ based on arbitrary weights $\{w_1, w_2, w_3, w_4\} > 0$. Then the only possible total scores are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} w_1 + w_2 + w_3 + w_4 \\ w_1 + w_2 + w_3 + 0 \\ w_1 + w_2 + 0 + 0 \\ w_1 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 \end{bmatrix}$$

The largest X_T will be largest for any choice of $w_4 > 0$
The 2nd largest will be so regardless of choice of $w_3 > 0$
and so on by induction to any AST scale.

This is non-parametric. Parametric models are next.

Absolute Simplex Model Structures

Mean Parameterization

Parameters are: $\mu_1, \mu_2, \ldots, \mu_p$

Raw 2nd Moment Matrix (MM) $\Sigma_m = \Sigma + \mu\mu' = \Sigma_m(\mu)$

Regression Model $P(y) = \mathbf{X}\beta_\mu$

Variation Parameterization

Parameters are: $\nu_1, \nu_2, \ldots, \nu_p \; [\nu_i = \sigma_i / \mu_i]$

Coefficient of Variation (CV) Model $\Sigma_\nu = D_\mu^{-1} \Sigma D_\mu^{-1} = \Sigma_\nu(\nu)$

Regression Model $O(y) = \mathbf{X}\beta_\nu$

A Distance Property

Distances based on MM and CV are unidimensional. Let A be any $p \times p$ pd matrix from moments of X_i and X_j with $0 < \alpha_i < \alpha_j < \infty$ such that $a_{ij} = \alpha_i$ if $i < j$. Also, let Δ have elements $\delta_{ij} = a_{ii} + a_{jj} - 2a_{ij}$. Then we have

$$A = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \alpha_4 - \alpha_1 \\ \alpha_2 - \alpha_1 & 0 & \alpha_3 - \alpha_2 & \alpha_4 - \alpha_2 \\ \alpha_3 - \alpha_1 & \alpha_3 - \alpha_2 & 0 & \alpha_4 - \alpha_3 \\ \alpha_4 - \alpha_1 & \alpha_4 - \alpha_2 & \alpha_4 - \alpha_3 & 0 \end{bmatrix}$$

With $h < i < j$, $\delta_{hj} = \delta_{hi} + \delta_{ij}$ are additive, i.e., the variables are unidimensional.

An Ordering Property

Population moment matrices in AST theory have an important property:

The variables (items) can be ordered by the magnitude of column sums and/or standard deviations of the moment matrix.

Reminder:

$$A = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}$$

This result is useful for ordering variables based on sample moment matrices.

Mean Parameterization

Let $\mu_1 < \mu_2 ... < \mu_p$. In an AST population the moment matrix $\Sigma_m = \Sigma + \mu\mu' = \Sigma_m(\mu)$ has a 1-parameter per item structure as shown above:

$$\Sigma_m = \begin{bmatrix} \mu_1 & \mu_1 & \mu_1 & \mu_1 & \mu_1 \\ \mu_1 & \mu_2 & \mu_2 & \mu_2 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 & \mu_3 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{bmatrix}$$

Items showing such a structure are unidimensional.
Example of 10 persons responding to 4 items

Person↓	Item→	1	2	3	4
1		1	1	1	1
2		1	1	1	1
3		1	1	1	0
4		1	1	1	0
5		1	1	1	0
6		1	1	0	0
7		1	1	0	0
8		1	1	0	0
9		1	0	0	0
10		0	0	0	0
$\bar{X}_i$		.9	.8	.5	.2

Clearly, e.g. $\frac{1}{10}\left(\sum X_3 X_4 = \sum X_2 X_4 = \sum X_1 X_4 = \sum X_4\right) = \bar{X}_4 = .2$

A consistent sample estimator of $\Sigma_m = \Sigma + \mu\mu' = \Sigma_m(\mu)$ is $S_m = S + \bar{X}\bar{X}'$. The sampling distribution of sample moments is known in structural equation modeling (e.g., Satorra, 1992; Yuan & Bentler, 1997).

Thus, the structural model $\Sigma_m(\mu)$ can be fit to S_m .

Asymptotically distribution-free (minimum chi-square, minimum distance) approaches give estimates, standard errors, and a χ^2 goodness-of-fit model test. More stable estimators in small samples such as normal theory (e.g. ML) are consistent, but robust test statistics and standard errors (e.g., Satorra & Bentler, 1994; Yuan & Bentler, 2010) must be used.

An example moment matrix S_m for male sexual behavior based on N=175 is:

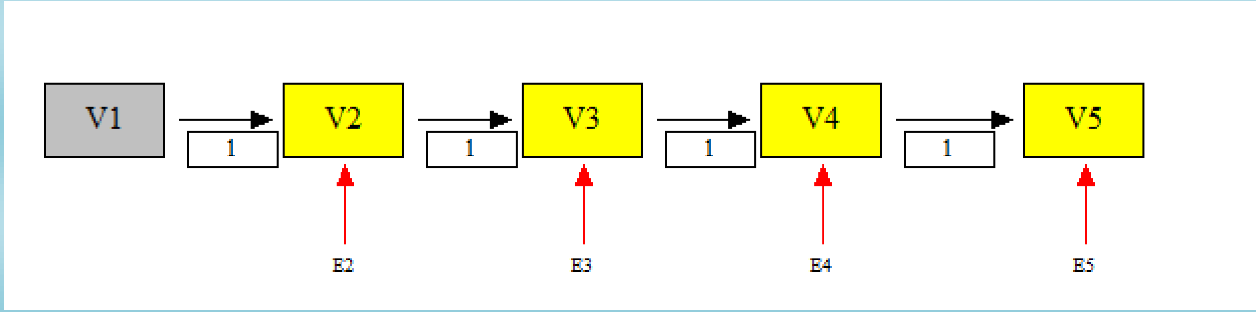
	V1	V2	V3	V4	V5	V6	V7	V8
V1	.891	.771	.697	.583	.566	.497	.394	.377
V2	.771	.789	.686	.577	.566	.491	.377	.377
V3	.697	.686	.709	.571	.531	.497	.383	.377
V4	.583	.577	.571	.594	.526	.463	.371	.383
V5	.566	.566	.531	.526	.577	.429	.360	.366
V6	.497	.491	.497	.463	.429	.509	.366	.366
V7	.394	.377	.383	.371	.360	.366	.411	.337
V8	.377	.377	.377	.383	.366	.366	.337	.389

Some Specific Model Structures

1. Σ_m a symmetric matrix with equality constraints.
2. $\Sigma_m = TD_{\mu diff}T'$ with T a lower triangular 1's matrix and

$$D_{\mu diff} = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 - \mu_1 & & \\ & & \cdots & \\ & & & \mu_p - \mu_{p-1} \end{bmatrix}$$

3. Autoregressive approach, e.g.,



4. Covariance structure approach. Since $\Sigma_m = \Sigma + \mu\mu'$ it follows that $\Sigma = \Sigma_m - \mu\mu'$. To run, just add one factor to above model with variance -1, plus constraints. In setup 2, $\mu = Td_{\mu diff}$ with $d_{\mu diff} = D_{\mu dif}1$. Or $\Sigma = T(D_{\mu diff} - d_{\mu diff}d_{\mu diff}')T'$.

5. Precision matrix approach. Model is for Σ_m^{-1}, the inverse of Σ_m. Let

$$D = D_{\mu diff}^{-1} = \begin{bmatrix} d_1 = \frac{1}{\mu_1} & & & \\ & d_2 = \frac{1}{(\mu_2 - \mu_1)} & & \\ & & \cdots & \\ & & & d_p = \frac{1}{(\mu_p - \mu_{p-1})} \end{bmatrix}$$

Then Σ_m^{-1} is tridiagonal, a function of the same p parameters. Here is a 4-item example:

$$\Sigma_m^{-1} = \begin{bmatrix} d_1 + d_2 & -d_2 & 0 & 0 \\ -d_2 & d_2 + d_3 & -d_3 & 0 \\ 0 & -d_3 & d_3 + d_4 & -d_4 \\ 0 & 0 & -d_4 & d_4 \end{bmatrix}$$

In all the above models, restrictions can be freed for more general models. An example is lag-2 effects in the autoregressive structure.

Structural Model for Transformed Moment Matrix

Instead of modeling Σ_m (& later, a related matrix Σ_ν), we can transform the data and hence the moment matrices.

Let $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, then $T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

Transform items as $Y = T^{-1}X$, and accordingly transform S_m and Σ_m e.g., $\tilde{S}_m = T^{-1}S_m T^{-1\prime}$. A simpler model structure results: $\tilde{\Sigma}_m = T^{-1}\Sigma_m T^{-1\prime} = T^{-1}(TD_{\mu diff}T')T^{-1\prime} = D_{\mu diff}$, a diagonal matrix.

Absolute Quasi-simplex Models

Model variants that allow excess variability are natural. They allow the diagonals of Σ_m to differ from those of the off-diagonals. A specific example is $\Sigma_m = TD_{\mu diff}T' + \Psi_m$, where Ψ_m is a diagonal matrix. Unlike factor analysis, one parameter in Ψ_m must be fixed, so the quasi-simplex AST model has $(2p - 1)$ parameters.

With appropriate adjustments, all the results for an AST simplex are relevant to the latent AST quasi-simplex, such as $(\Sigma_m - \Psi_m) = TD_{\mu diff}T'$. The parameters now refer items to a latent continuum.

Information in Higher Moments

3rd –order item products also provide information about model parameters for gains in efficient estimation. The earlier 10x4 data yield 10x10 moments based on 4 parameters:

Item	1	2	1×2	3	1×3	2×3	4	1×4	2×4	3×4
1	.9									
2	.8	.8								
1×2	.8	.8	.8							
3	.5	.5	.5	.5						
1×3	.5	.5	.5	.5	.5					
2×3	.5	.5	.5	.5	.5	.5				
4	.2	.2	.2	.2	.2	.2	.2			
1×4	.2	.2	.2	.2	.2	.2	.2	.2		
2×4	.2	.2	.2	.2	.2	.2	.2	.2	.2	
3×4	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2

Data-based Interval Scale Scores

Since X_T completely orders the distribution of a uni-dimensional absolute simplex, it can be used to get the empirical cumulative distribution function (CDF) of the trait.

Given the CDF, we can use the inverse normal distribution function to compute z-scores.

This produces an interval scale if we are correct that the trait is normally distributed.

00000
00001
00011
00011
00011
00111
00111
00111
01111
11111

Ordered Total Scores Generate the CDF of the Scale

% of subjects below a pattern
= % of subjects below total score

Pattern	Score X_T	%below	CDF
11111	5	90	1.00
01111	4	80	.90
00111	3	50	.80
00011	2	20	.50
00001	1	10	.20
00000	0	0	.10

Regression Estimation

The proportion of the population below a given total score is $P(y) = prob(Y < y)$. We have

$$P(y) = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$$

predicts the $\text{CDF}_{\text{BELOW}}$ exactly with $R^2 = 1.0$. The μ parameters can be obtained from the β via $\beta_p = 1 - \mu_p$ and $\beta_i = \mu_{i+1} - \mu_i$.

In practice we get an estimate $\tilde{y}$ of $P(y)$ and run the regression $\tilde{y} = \mathbf{X}\beta + e$ to get the estimator $\hat{\beta}$ and the associated statistics.

The closeness of $\hat{R}^2$ to 1.0 provides a measure of the validity of the model. Robust standard errors are available.

Adding items with $\beta_i > 0$, $P(y)$ becomes continuous as $p \to \infty$, i.e., in a "universe" of items. If also $n \to \infty$, then $P(y)$ approaches the population/universe trait CDF.

Notice that *the population distribution function can be approximated by* $\mathbf{X}\hat{\beta}$ *without the use of any norms*. The item parameters carry this information. Also, $\mathbf{X}\hat{\beta}$ is a *formative* measure – the trait arises from the item responses.

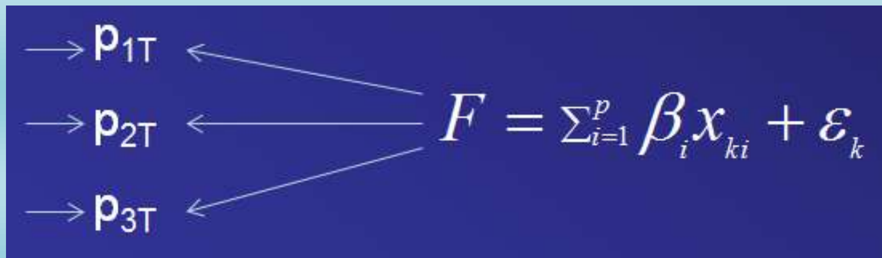
If the population distribution is normal, normal *z*-scores can be obtained through the inverse normal CDF. In practice, it can be desirable to compute an estimate of $P(y)$ independently of $\mathbf{X}$ for the regression.

Mimic Model Estimation

With many items, items can be grouped into sets, each with a full range of item content, item means, and with its own total score X_{1T}, X_{2T}, …

The several X_{1T}, X_{2T}, … can yield several $P(y)$ - proportions below - such as p_{1T}, p_{2T}, …

A latent factor F can be created and a mimic model used in place of regression estimation. This means $\mathbf{X}\hat{\beta}$ estimates the latent trait F proportion below.



Uncontaminated Bivariate Regression

(with a normality assumption)

1. Compute $S_m = S + \bar{X}\bar{X}'$
2. Estimate $\hat{\mu}$ based on S_m
3. Compute $\hat{\beta}$ based on $\hat{\mu}$
4. Compute $\hat{P}(y) = X\hat{\beta}$ and estimated CDF
5. Compute $\hat{z}$ from estimated CDF using inverse normal
6. Compute bivariate regression of an "experimentally independent" observed score on the predicted $\hat{z}$ from the model

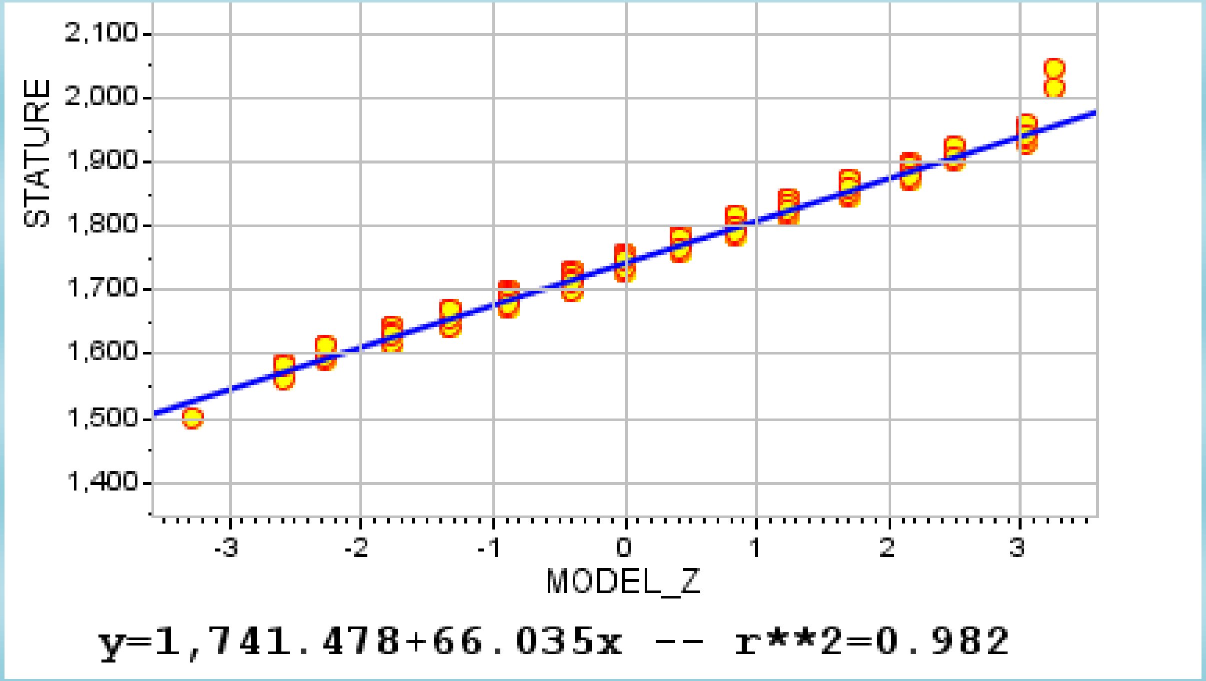
The following example is not pure according to #6, but it shows how well the AST model can recover quantitative data.

Male Stature (Height) in cm (n= 1774)

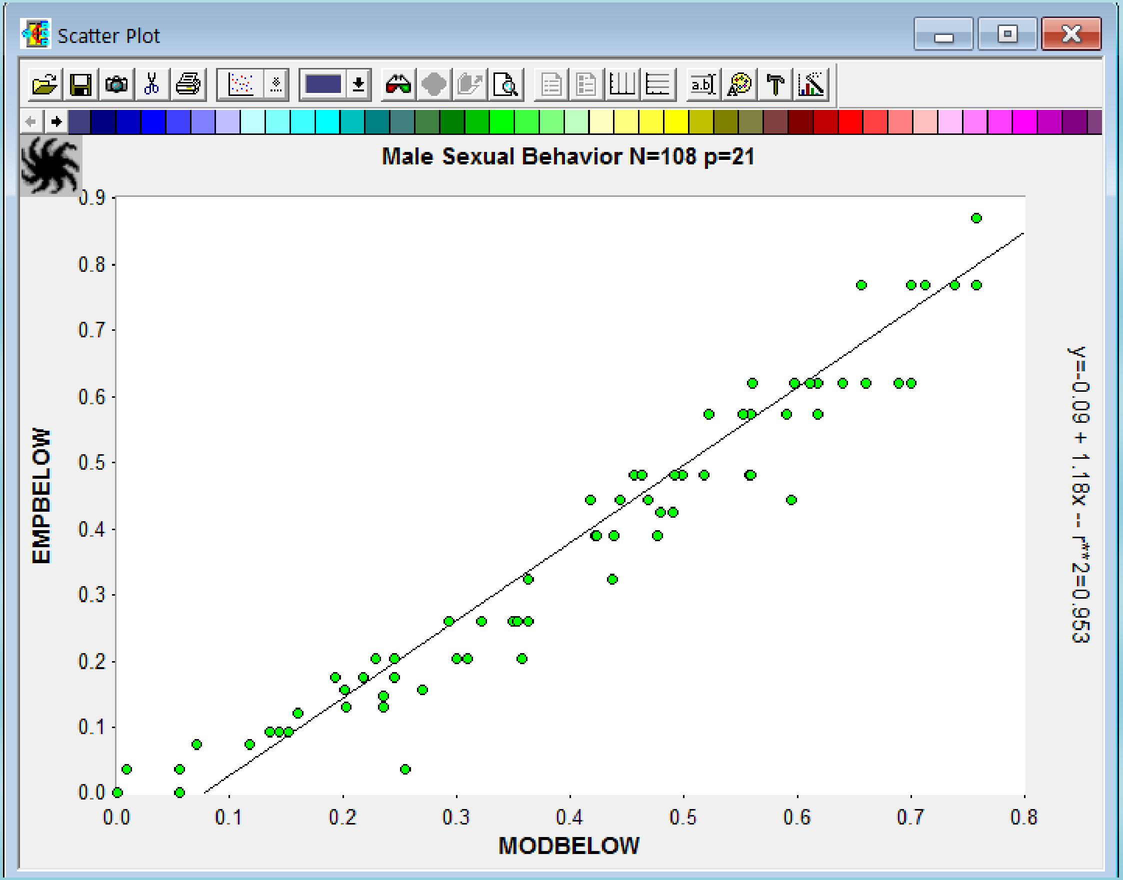
15 artificial Guttman items created from national data.

AST model fitted, z-scores obtained, and height predicted.

Extreme binary data was all 1's, or all 0's – no Bayes.



Male Sexual Behavior
21 parameter AST model – Distribution free, no z



Variation Parameterization

The Coefficient of Variation (CV) matrix $\Sigma_\nu = D_\mu^{-1} \Sigma D_\mu^{-1}$ with $\sigma_{\nu i}^2 = \sigma_i^2 / \mu_i^2$ and $\sigma_{\nu ij} = \sigma_{ij} / \mu_i \mu_j$ in a strict AST population has diagonal elements $\sigma_{\nu i}^2$ and off-diagonal elements $\sigma_{\nu i}^2$ if $i < j$.

This is a *p*-parameter model as before, with patterns of equalities and inequalities similar to that shown for the means. Thus it also gives a unidimensional representa-tion. If there is additional variation, we have the quasi simplex $\Sigma_\nu^* = \Sigma_\nu + \Psi_\nu$ with Ψ_ν diagonal.

This was the 1971 parameterization.

Here’s a small CV matrix for 4 items:

.111	.111	.111	.111
.111	.250	.250	.250
.111	.250	1.000	1.000
.111	.250	1.000	4.000

The CV matrix has the same pattern as the moment matrix. Unidimensionality and the same six model types (with variation parameters instead of means) can be used to estimate and test the model. There is also a 7th method. Furthermore, the absolute quasi-simplex with added diagonal variation exists for all variants.

Seven CV Models

(1) a patterned matrix with equality constraints and parameters $v_1^2, v_2^2, \ldots, v_p^2$;

(2) the structure $\Sigma_v = TD_{vdiff}T'$, where *T* is a lower triangular matrix with 1.0 elements and D_{vdiff} is a diagonal matrix with diagonal elements $v_1^2, v_2^2 - v_1^2, \ldots, v_p^2 - v_{p-1}^2$;

(3) an autoregressive model based on equations $X_1 = \xi_1,\ X_2 = \xi_1 + \xi_{21}, \ldots, X_j = \xi_i + \xi_{ji}, \ldots, X_p = \xi_{p-1} + \xi_{p,p-1}$ with moment parameters of ξ_1 and the ξ_{ji} as $v_1^2, v_2^2 - v_1^2, \ldots, v_p^2 - v_{p-1}^2$.

(4) tri-diagonal precision matrix Σ_v^{-1}

(5) transformed variables and moments method

(6) covariance structure. Let $D_{\tilde{v}}$ be a diagonal matrix that contains $(v_i^2+1)^{-1}$ as its i^{th} diagonal element. Then, a covariance structure model for an absolute simplex is given by $\Sigma = D_{\tilde{v}} T D_{vdiff} T' D_{\tilde{v}}$.

A 7th Structure: Simplex Correlation Structure

The correlation of two AST variables with variation parameters $v_i < v_j$ is given by the ratio $\rho_{ij} = v_i / v_j$. This is similar to Guttman's (1954) simplex correlation matrix, but AST variation parameters are absolute and cannot be rescaled. Hence AST simplex$\neq$Guttman simplex, though they are similar.

CV Estimation Methods

Let $D_{\bar{X}}$ be a diagonal matrix that contains sample means. Then a consistent estimator of the CV matrix Σ_v is $S_v = D_{\bar{X}}^{-1} S D_{\bar{X}}^{-1}$.

Hence we may fit $\Sigma_v(\nu)$ to S_v to get $\hat{\nu}$. Bentler (2009) developed the asymptotic distribution of S_v as a delta method function of the known asymptotic distribution of means and covariances. Then the usual wide variety of standard SEM estimators, standard errors, model tests, etc. become available. See also Boik and Shirvani (*Statistical Methodology,* 2009).

Regression Estimation

The proportion of the population below a given score X_T, $P(y) = prob(Y < y)$, was described before. Now we consider person odds-below $O(y) = P(y)/\{1 - P(y)\}$. Then we have

$$O(y) = \mathbf{X}\beta_\nu$$

where $\beta'_\nu = (\nu_1^2, \nu_2^2 - \nu_1^2, \dots, \nu_p^2 - \nu_{p-1}^2)$.

An illustrative example of person odds-below in relation to variation parameters can be seen from the example given previously.

Person	Items				Total Score X_T	Prop. Persons Below	Odds Persons Below
1	1	1	1	1	4	.80	4.0
2	1	1	1	1	4	.80	4.0
3	1	1	1	0	3	.50	1.0
4	1	1	1	0	3	.50	1.0
5	1	1	1	0	3	.50	1.0
6	1	1	0	0	2	.20	.25
7	1	1	0	0	2	.20	.25
8	1	1	0	0	2	.20	.25
9	1	0	0	0	1	.10	.11
10	0	0	0	0	0	0	0
$\bar{X}_i$	.9	.8	.5	.2			
s_i^2	.09	.16	.25	.16			
$s_i^2 / \bar{X}_i^2$	.11[a]	.25	1.0	4.0	←Variation Parameters		

In practice, we can estimate $O(y)$ from X_T and add an error term to the model. A high $\hat{R}^2$ is desirable.

To get a latent variable model, as before, we need multiple total scores X_{1T}, X_{2T}, … that yield several $P(y)$ - proportions below and, in turn, several odds below. Then we put a factor F behind these, and use a mimic model to predict F.

This gives an estimated latent trait CDF_{BELOW} for each individual, and hence provides the latent CDF. If normality is assumed, a *z*-score can be obtained.

Summary

- Walker, Guttman, Loevinger had good ideas
- Guttman scales are defined on response patterns and hence are hard to incorporate into standard statistical models and methods
- Absolute simplex theory was published in 1971 for CV matrices, but structural modeling did not exist and no formal statistical machinery was available
- AST theory has now been extended to a wide variety of model types with standard statistical methodologies
- Evaluation of these methodologies and comparison to existing scaling methodologies such as item response theory, classical test theory, etc. remains to be done.

Additional References

Boik, R. J., & Shirvani, A. (2009). Principal components on coefficient of variation matrices. *Statistical Methodology*, *6*, 21-46.

Satorra, A. (1992). Asymptotic robust inferences in the analysis of mean and covariance structures. *Sociological Methodology*, *22*, 249-278.

Satorra, A., & Bentler, P. M. (1994). Corrections to test statistics and standard errors in covariance structure analysis. In A. von Eye & C. C. Clogg (Eds.), *Latent variables analysis: Applications for developmental research* (pp. 399-419). Thousand Oaks, CA: Sage.

Yuan, K. –H., & Bentler, P. M. (1997). Mean and covariance structure analysis: Theoretical and practical improvements. *Journal of the American Statistical Association*, *92*, 767-774.

Yuan, K. –H., & Bentler, P. M. (2010). Two simple approximations to the distributions of quadratic forms. *British Journal of Mathematical and Statistical Psychology*, *63*, 273-291.

That's All, Thank You