Dividend maximization under regime switching and incomplete information

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Our aim is to study the valuation problem of an (insurance) company.

How can we define the value of a company?

De Finetti (1957)

$$\sup_u \mathbb{E}\left(\int_0^\tau e^{-\delta t} u_t \, dt\right)$$

Connections to optimal consumption

Related to pure optimal consumption problem of an economic agent: maximize the expected value of accumulated discounted consumption.

Another variant is maximizing the expected discounted utility of consumption

- Hubalek & Schachermayer (2004)

$$\sup_c \mathbb{E}\left(\int_0^\tau e^{-\delta t} U(c_t)\, dt\right) ,$$

or today's expected utility of future consumption

- Grandits, Hubalek, Schachermayer & Žigo (2007)

$$\sup_c \mathbb{E}\left(U\left(\int_0^\tau e^{-\delta t} c_t\, dt\right)\right) .$$

$$\sup_u \mathbb{E}\left(\int_0^\tau e^{-\delta t} u_t \, dt\right)$$

$$V = \sup_u \mathbb{E}\left(\int_0^\tau e^{-\delta t} u_t \, dt\right)$$

We do not study utility – we study pure cashflows!

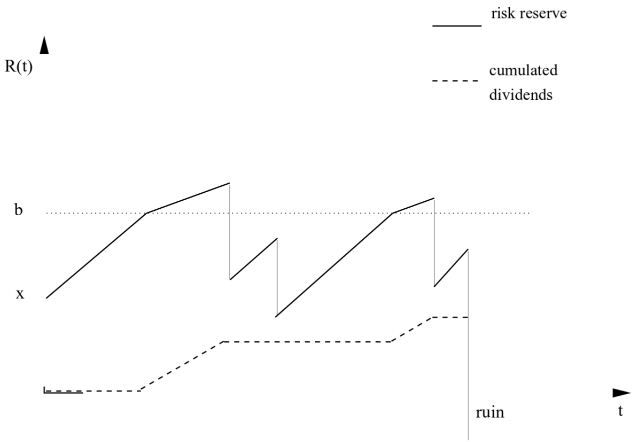
These cashflows are paid from the underlying surplus X.

Tradeoff between paying a high dividend rate and ruin.

Find the optimal dividend strategy u and the optimal value function V.
$\rightarrow$ Risk measure!

Optimal dividend strategy

Threshold strategy



Source: Albrecher & Thonhauser (2009)

Asmussen & Taksar (1997)

Diffusion model

Surplus of the insurance company

$$dX_t = (\mu - u_t)dt + \sigma dW_t \,,$$

where $u_t \in [0, K] \ \forall t$.

Solution to

$$\mu v_x + \frac{\sigma^2}{2} v_{xx} - \delta v + \sup_{u \in [0,K]} (u(1 - v_x)) = 0$$

is the optimal value function.

Asmussen & Taksar (1997)

$$V(x) = \begin{cases} a_1 \exp(\alpha_1(x-\bar{b})) + a_2 \exp(-\alpha_2(x-\bar{b})), & x < \bar{b} \\ b_2 \exp(-\beta_2(x-\bar{b})) + \frac{K}{\delta}, & x \geq \bar{b}\,, \end{cases}$$

and

$$\alpha_1 = \frac{1}{\sigma^2}\left(-\theta + \sqrt{\theta^2 + 2\sigma^2\delta}\right), \qquad \alpha_2 = \frac{1}{\sigma^2}\left(\theta + \sqrt{\theta^2 + 2\sigma^2\delta}\right),$$

$$\beta_2 = \frac{1}{\sigma^2}\left(\theta - K + \sqrt{(\theta-K)^2 + 2\sigma^2\delta}\right), \qquad a_1 = \frac{\alpha_2(\frac{K}{\delta} - \frac{1}{\beta_2}) + 1}{\alpha_1 + \alpha_2},$$

$$a_2 = \frac{\alpha_1(\frac{K}{\delta} - \frac{1}{\beta_2}) - 1}{\alpha_1 + \alpha_2}, \qquad b_2 = \begin{cases} -\frac{1}{\beta_2} & \bar{b} > 0 \\ -\frac{K}{\delta} & \bar{b} = 0\,, \end{cases}$$

$$\bar{b} = \left(\frac{1}{\alpha_1 + \alpha_2}\log\left(-\frac{a_1}{a_2}\right)\right)_+ \,.$$

Threshold strategies are optimal.

Sotomayor & Cadenillas (2012)

Regime switching

Different regimes represent different phases of the economy.

Surplus of the insurance company

$$dX_t = (\mu(Y_t) - u_t)dt + \sigma(Y_t)dW_t \,,$$

where $u_t \in [0, K]$.

- $(Y_t)_{t\geq 0}$ is a Markov chain with M-dimensional state space.
- We know the initial state $Y_0 = e_i$.

Sotomayor & Cadenillas (2012)

Solution to

$$\frac{\sigma_i^2}{2} v_{xx}(x,i) + \mu_i v_x(x,i) - \delta v(x,i) + \sup_{u \in [0,K]} \left[u\left(1 - v_x(x,i)\right) \right] = \lambda_i v(x,i) - \sum_{j \neq i} q_{ij} v(x,j)$$

is the optimal value function.

Again a threshold strategy with - at least for each state - constant threshold level is optimal.

Note: Asmussen & Taksar is special case with $Q \equiv 0$.

Further contributions (incomplete)

- Shreve, Lehoczky and Gaver
- Radner and Shepp
- Paulsen
- Jiang and Pistorius
- with jumps: Loeffen
- C-L model: Albrecher and Thonhauser, Scheer and Schmidli, ...
- ...

All of these assume complete information → we: incomplete information.

Our model for the surplus process

Brownian motion with unobservable drift, absorbed when hitting zero.

- Insurance context: leftover uncertainty of diffusion approximation
- Profitability of business activities
- Economic environment

- Brownian motion with unobservable drift that might shift.
- Different phases of the economy
- Reasonable, since we consider a possibly long time horizon
- Realistic not to know the current drift.

Hidden Markov models are frequently used in mathematical finance.

Dividend payment directly influences the underlying surplus process.

The ingredients of the model I

- Surplus process of the insurance company

$$dX_t = (\mu(Y_t) - u_t)dt + \sigma dB_t = dZ_t - dL_t\ ,$$

$X_0 = x$, $Z_0 = z = x$.

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$X_0 = x$, $Z_0 = z = x$.

- $Y = (Y_t)_{t \geq 0}$ is an M-state Markov chain with known generator matrix $Q = (q_{ij})_{i,j=1}^M$;
- $Y = (Y_t)_{t \geq 0}$ is an M-state Markov chain with known generator matrix $Q = (q_{ij})_{i,j=1}^M$;
- Bayesian case: $Q \equiv 0$ and $\mu(Y_t) \equiv \mu$.
- $\mu(Y_t) = \mu_i$, if $Y_t = e_i$;
- We know the initial state $Y_0 = e_i$.

The ingredients of the model II

- Cumulated dividend process up to time t

$$L_t = \int_0^t u_s \, ds \,,$$

with $u_s \in [0, K] \; \forall s$.

- Time of ruin $\tau := \inf\{t \geq 0 | X_t \leq 0\}$.

- Observation filtration

$$\mathcal{F}_t^Z = \sigma(Z_s, 0 \leq s \leq t) \,.$$

$\rightarrow$ Hidden Markov model

Filtering theory I

Replace $\mu(Y_t)$ by an estimator $(\nu_t)_{t\geq 0}$

$$\nu_t = \mathbb{E}(\mu(Y_t)|\mathcal{F}_t^Z)\,.$$

Estimator:

$$\nu_t = \sum_{i=1}^{M} \mu_i \pi_i(t)\,,$$

where $\pi_i(t) = \mathbb{P}(Y_t = e_i|\mathcal{F}_t^Z)$.

Filtering theory II

Shiryaev/Wonham Filter (optimal!): estimates probability to be in state i at time t:

$$\pi_i(t) = p_i + \int_0^t \sum_{j=1}^{M} q_{ji}\pi_j(s)\, ds + \int_0^t \pi_i(s) \frac{\mu_i - \sum_{i=1}^{M} \mu_i \pi_i(s)}{\sigma}\, dW_s\,.$$

$$\pi_i(t) = p_i + \int_0^t \sum_{j=1}^{M} q_{ji}\pi_j(s)\, ds + \int_0^t \pi_i(s) \frac{\mu_i - \sum_{i=1}^{M} \mu_i \pi_i(s)}{\sigma}\, dW_s\,.$$

Innovation process

$$W_t = \frac{\mu_t - \nu_t}{\sigma} dt + B_t$$

In the Bayesian case: could also use Bayes' rule.

Underlying process I

Finally, we arrive at

$$X_t = x + \int_0^t (\nu_s - u_s)\; ds + \sigma W_t\,,$$

$$\pi_i(t) = p_i + \int_0^t \sum_{j=1}^{M} q_{ji}\pi_j(s)\, ds + \int_0^t \pi_i(s) \frac{\mu_i - \sum_{i=1}^{M} \mu_i \pi_i(s)}{\sigma}\, dW_s\,,$$

$i \in \{1, \ldots, M-1\}$.

Since $\pi_M(t) = 1 - \sum_{i=1}^{M-1} \pi_i(t)$, we end up with M-dimensions.

Full information, but cost: extra dimensions.

Underlying process II

To keep numerics tractable, choose $M = 2$

$$dX_t = \left(\sum_{i=1}^{M} \mu_i \pi_i(t) - u_t\right) dt + \sigma dW_t \,,$$

$$d\pi_1(t) = \sum_{j=1}^{M} q_{ji}\pi_j(t)\, dt + \pi_1(t) \frac{\mu_1 - \sum_{i=1}^{M} \mu_i \pi_1(t)}{\sigma}\, dW_t \,,$$

and replace π_1 by ν:

$$\begin{aligned} dX_t &= (\nu_t - u_t)\ dt + \sigma dW_t \,, \\ d\nu_t &= (q_{21}(\mu_1 - \nu_t) + q_{11}(\nu_t - \mu_2))\ dt + \frac{(\mu_1 - \nu_t)(\nu_t - \mu_2)}{\sigma}\, dW_t \,. \end{aligned} \tag{1}$$

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Maximize the expected discounted future dividend payments until the time of ruin.

Find optimal value function

$$V(x,\upsilon) = \sup_{u \in A} J^{(u)}(x,\upsilon) = \sup_{u \in A} \mathbb{E}_{x,\upsilon}\left(\int_0^\tau e^{-\delta t} u_t \, dt\right)$$

and optimal control $u \in A$.

Theorem (Bellman principle)

For every bounded stopping time η we have

$$V(x,\upsilon) = \sup_{u\in A} \mathbb{E}_{x,\upsilon} \left(\int_0^{\tau\wedge\eta} e^{-\delta t} u_t \, dt + e^{-\delta(\tau\wedge\eta)} V(X_{\tau\wedge\eta}, \nu_{\tau\wedge\eta}) \right) .$$

Apply Itô's formula to $V(X_{\tau\wedge\eta}, \nu_{\tau\wedge\eta})$ to get the Hamilton-Jacobi-Bellman (HJB) equation.

We have to verify the solution!

Hamilton-Jacobi-Bellman (HJB) equation

$$(\tilde{\mathcal{L}} - \delta)v + \sup_{u \in [0,K]} (u(1 - v_x)) = 0\,, \tag{3}$$

where

$$\begin{aligned}\tilde{\mathcal{L}}V = & \upsilon V_x + \left(q_{21}(\mu_1 - \upsilon) + q_{11}(\upsilon - \mu_2)\right) V_\upsilon + (\mu_1 - \upsilon)(\upsilon - \mu_2) V_{x\upsilon} \\ & + \frac{1}{2\sigma^2}(\mu_1 - \upsilon)^2(\upsilon - \mu_2)^2 V_{\upsilon\upsilon} + \frac{1}{2}\sigma^2 V_{xx}\,.\end{aligned}$$

The supremum in (3) is

$$\sup_{u \in A}(u(1 - v_x)) = \begin{cases}(1 - v_x)\,K, & v_x \leq 1 \\ 0, & v_x > 1\,.\end{cases}$$

The HJB equation is a second-order degenerate elliptic PDE.

Boundary conditions (BC)

- For the x-component the boundary conditions are

$$
\begin{aligned}
v(0, \upsilon) &= 0\,, \\
v(B, \upsilon) &= \frac{K}{\delta} \text{ for } B \to \infty\,.
\end{aligned}
$$

- Bayesian case: the ones for $\upsilon \to \theta_i$, $i = 1, 2$ are obtained by solving the optimal control problem for known deterministic drift, as has been computed by Asmussen & Taksar (1997).
- Markov switching case: cannot use solution to deterministic problem in the directions of the filter. $\to$ no boundary conditions available!

Characterization of the solution I

Viscosity solution characterization of the optimal value function.

Theorem (S.)

V is the unique bounded viscosity solution of (3) with boundary conditions (BC).

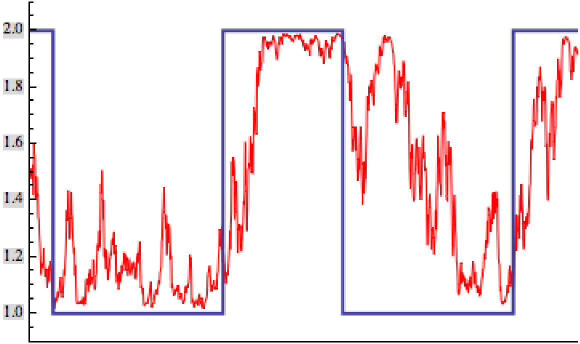
Therefore prove

- V is viscosity sub- and supersolution
- Comparison result

Characterization of the solution II

In the Markov switching case: no boundary conditions in the direction of the filter.

But: the filter does not reach the boundary!



$\rightarrow$ uniqueness can still be proven!

What about smooth solutions?

Theorem (S.)

Let $v(x, \vartheta)$ be a viscosity supersolution of (3) and let $v \in C^2$ almost everywhere.
Then $V \leq v$.

Remark

Suppose one can construct a strategy $\tilde{u}$ such that $J^{(\tilde{u})}$ is a supersolution with $J^{(\tilde{u})} \in C^2$ almost everywhere.
Then the theorem implies that $J^{(\tilde{u})} = V$ and thus $\tilde{u}$ is the optimal strategy.

- We start with a simple (threshold) strategy:

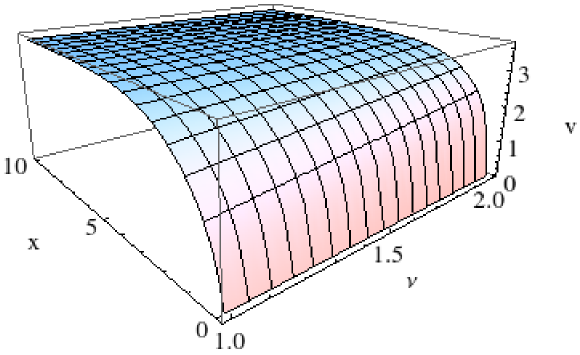
$$u^{(0)}(x, \upsilon) = K 1_{\{x \geq b_0(\upsilon)\}} \,.$$

- Policy iteration
- For a given Markov strategy $u^{(k)}$ we solve

$$(\tilde{\mathcal{L}}^G - \delta)V + u^{(k)}(1 - \mathcal{D}_x^G V) = 0 \,.$$

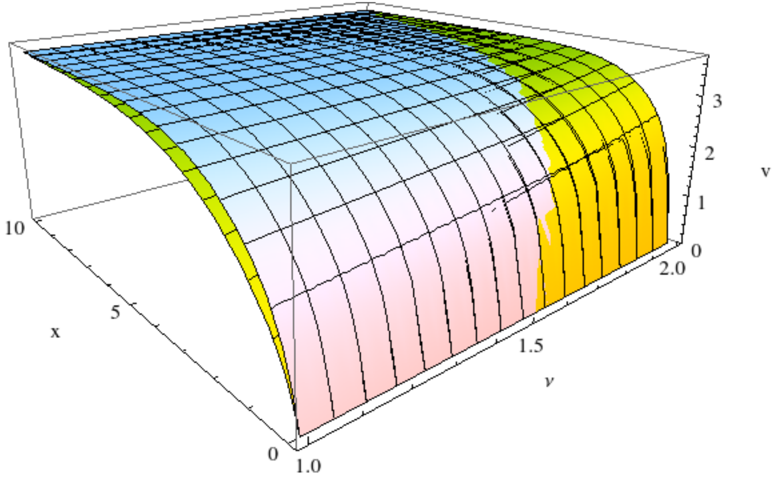
- Maximization yields $u^{(k+1)}(x, \upsilon) = K 1_{\{\mathcal{D}_x^G V(x,\upsilon) \leq 1\}}$.
- The iteration stops as soon as $u^{(k+1)} \approx u^{(k)}$.

Value function



$\sigma = 1$, $\mu_1 = 1$, $\mu_2 = 2$, $\delta = 0.5$, $-q_{11} = 0.25$, $q_{21} = 0.5$, $K = 1.8$

Value Bayes vs. Switching

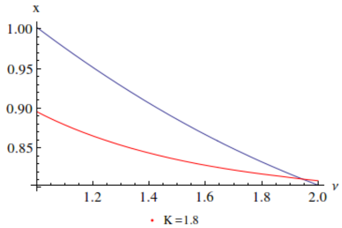


(Bayes → yellow/green, Switching → pink/blue)

Dividend policy

In the full-information setup $\rightarrow$ threshold strategy optimal.

Numerical solution (Bayes $\rightarrow$ red, Switching $\rightarrow$ blue)



Threshold strategy

$$u_t = K \cdot 1_{[b(\nu_t),\infty)}(X_t)$$

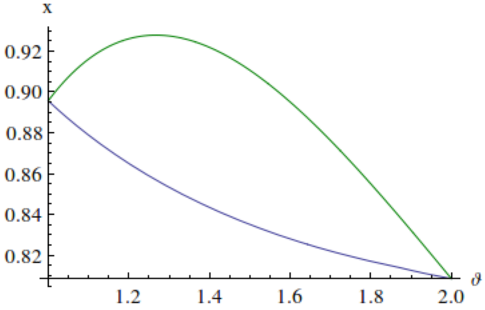
with sufficiently smooth threshold level b.

The same type of strategies is obtained in the jump-diffusion case.

Bayesian vs. deterministic case

Compare threshold level of the Bayesian case to threshold level of the deterministic case for varying μ.

(Bayes $\rightarrow$ blue, Classical $\rightarrow$ green)



Optimality of threshold strategies

Let $J^{(b)}$ be the viscosity solution to

$$(\mathcal{L}-\delta)J^{(b)} + K(1 - J_x^{(b)})1_{\{x \geq b(\upsilon)\}} = 0$$

with the same boundary conditions as for the HJB equation.

Theorem (Leobacher, S., Thonhauser)

Let a threshold level $b : [\mu_1, \mu_2] \longrightarrow [0, \infty)$ *exist with* $u^b \in A_M$ *and* $(J_x^{(b)}(x, \upsilon) \leq 1 \Leftrightarrow x \geq b(\upsilon))$ *in the viscosity sense.*

Then $J^{(b)}$ *is a viscosity solution to the HJB equation, and since there is only one viscosity solution to the HJB equation,* $J^{(b)} = V$*.*

Are threshold strategies admissible?

Does system (2) have a solution?

$$dX_t = (\nu_t - K \cdot 1_{[b(\nu_t),\infty)}(X_t))dt + \sigma W_t \,,$$
$$d\nu_t = (q_{21}(\mu_1 - \nu_t) + q_{11}(\nu_t - \mu_2))\ dt + (\nu_t - \mu_1)(\mu_2 - \nu_t)dW_t \,,$$

$X_0 = x, \nu_0 = \upsilon.$

Theorem (Itô)

Assume that there exists a positive constant L such that for all $x, y \in \mathbb{R}^2$

$$\max\{\|f(x) - f(y)\|, \|g(x) - g(y)\|\} \leq L\|x - y\| .$$

Then there exists a unique solution x_t of equation

$$dx_t = f(x_t)dt + g(x_t)dW_t .$$

Our drift

$$\begin{pmatrix} \nu_t - K \cdot 1_{[b(\nu_t),\infty)}(X_t) \\ q_{21}(\mu_1 - \nu_t) + q_{11}(\nu_t - \mu_2) \end{pmatrix}$$

does not satisfy the condition!

Theorem (Veretennikov, Zvonkin)

- *f, g are bounded and measurable,*
- *g is Lipschitz,*
- *there exists a positive constant C such that for all $\lambda, x \in \mathbb{R}^2$*

$$\lambda^T g(x) g^T(x) \lambda \geq C|\lambda|^2 \,,$$

i.e., g is uniformly non-degenerate.

Then there exists a unique solution x_t of equation

$$dx_t = f(x_t)dt + g(x_t)dW_t \,.$$

Our diffusion coefficient

$$\begin{pmatrix} \sigma & 0 \\ \frac{1}{\sigma}(\mu_2 - \nu_t)(\nu_t - \mu_1) & 0 \end{pmatrix}$$

is degenerate!

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{4}$$

Theorem (Leobacher, S., Thonhauser)

Assume that

- σ *is Lipschitz,*
- $\sigma_{1j} \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$, *for* $j = 1, \ldots, d$,
- μ *satisfy a linear growth condition, i.e.,* $\|\mu(x)\| \le D_1 + D_2\|x\|$,
- $(\sigma\sigma^\top)_{11} \ge c > 0$ *for some constant* c *and for all* $x \in \mathbb{R}^d$.

The function μ *is allowed to be discontinuous along* $\{x_1 = 0\}$.

Assume further there exist functions $\mu^+, \mu^- \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$ *such that*

$$\mu(x_1, \ldots, x_d) = \begin{cases} \mu^+(x_1, x_2, \ldots, x_d) & \text{if} \quad x_1 > 0 \\ \mu^-(x_1, x_2, \ldots, x_d) & \text{if} \quad x_1 < 0 \end{cases}$$

Then there exists a unique global solution of equation (4).

Generalization

As a generalization we also proved the case where discontinuities appear not only along $\{x_1 = 0\}$, but also along some sufficiently regular hypersurface $\{x \in \mathbb{R}^d : f(x) = 0\}$.

The condition

$$\|\nabla f(x) \cdot \sigma(x)\|^2 \geq c > 0$$

replaces

$$(\sigma\sigma^\top)_{11} \geq c > 0\,.$$

This has a nice geometric interpretation: the diffusion component must not be parallel to the curve where the drift is discontinuous.

Back to our dividend problem

$$\mu(X_t,\nu_t) = \begin{pmatrix} \nu_t - K \cdot 1_{[b(\nu_t),\infty)}(X_t) \\ q_{21}(\mu_1 - \nu_t) + q_{11}(\nu_t - \mu_2) \end{pmatrix}$$
$$\sigma(X_t,\nu_t) = \begin{pmatrix} \sigma & 0 \\ \frac{1}{\sigma}(\mu_2 - \nu_t)(\nu_t - \mu_1) & 0 \end{pmatrix}$$
$$f(X_t,\nu_t) = X_t - b(\nu_t)$$

If the threshold level b is sufficiently regular, all conditions on the coefficients are fulfilled.

Thus, applying our theorem implies existence of a unique global solution of the system (2).

Threshold strategies are admissible!

Conclusion

- Dividend maximization problem in different hidden Markov models.
- Dividend policy of threshold type with threshold level depending on the estimate of the drift.
- Admissibility of threshold strategies.
- Existence and uniqueness of solutions for a class of SDEs with discontinuous drift and degenerate diffusion with applications in different branches of applied mathematics.

Announcement

10th IMACS Seminar on Monte Carlo Methods

July 6-10, 2015
Linz, Austria

www.mcm2015.jku.at

Thank you for your attention!