Utility-Risk Portfolio Selection

Harry Zheng
Imperial College

Vienna University of Economics and Business
December 4, 2014

This is a joint work with
Ryan KC Wong (University of Hong Kong and Imperial College) and
Philip SCP Yam (Chinese University of Hong Kong)

Mean-Variance and Utility Maximization

- Markowitz (1952) first introduced one-period mean-variance portfolio selection problem

$$\text{Minimize Var}(X) \text{ subject to } \mathbb{E}[X] \geq \mu.$$

- Samuelson (1969) and Merton (1971) extended the Markowitz's model to multiperiod and continuous time respectively, they used utility to assess the performance of portfolio, instead of using expected return and variance of wealth directly, and investigated utility maximization with stochastic control method.

$$\text{Maximize } \mathbb{E}[U(X_T)] \text{ subject to } X \text{ feasible wealth process.}$$

- Karatzas et al. (1987), Kramkov and Schachermayer (2003), and many others used martingale and convex duality method to solve utility maximization problem and to show existence of the optimal solution.

Risk Measures

- Risk is measured by

$$\mathbb{E}[D(\mathbb{E}[X_T] - X_T)],$$

where D is some nonnegative convex function measuring the deviation of terminal payoff X_T from its mean $\mathbb{E}[X_T]$.

- In Markowitz (1952) model, $D(x) = x^2$, variance.

- There are many other risk functions. For example,
  - Power risk function: $D(x) = \frac{|x|^{\rho+1}}{\rho+1}$ for a $\rho \geq 1$.
  - Variance: $\rho = 1$ for power risk function
  - Weighted variance: $D(x) = ax_+^2 + bx_-^2$ where $a \geq b > 0$.
  - Semi-variance: $D(x) = \frac{1}{2}x_+^2$, downside risk function.
  - Exponential risk function: $D(x) = e^x$.

- Other risk measures, such as VaR, CVaR, etc.

Model Setting

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space, over which $W(t) = (W_1(t), \ldots, W_m(t))^t$ denotes the $m-$dimensional standard Brownian motion and $\mathcal{F}_t := \sigma(W(s) : s \leq t)$.

- Suppose that the market has one riskless money account with price process $B(t)$ and m risky assets with the joint price process $S(t) := (S_1(t), \ldots, S_m(t))^t$ satisfying

$$
\begin{aligned}
dB(t) &= r(t)B(t)dt, \quad B(0) = b_0 > 0, \\
dS_k(t) &= \mu_k(t)S_k(t)\,dt + S_k(t)\sum_{j=1}^{m} \sigma_{kj}(t)dW_j(t), \quad S_k(0) = s_k > 0,
\end{aligned}
$$

  where $r(t)$ is the riskless interest-rate, $\mu_k(t)$ and $\sigma_k(t) := (\sigma_{k1}(t), \ldots, \sigma_{km}(t))$ are appreciation rate and volatility of k-th risky asset.

- Assume r, σ are uniformly bounded processes, σ is uniformly non-degenerate, i.e., $\sigma(t)\sigma(t)^t \geq \delta I$ for some $\delta > 0$, therefore $(\sigma(t))^{-1}$ exists for all t.

Controlled Wealth Process

- Let $\pi(t) := (\pi_1(t), \ldots, \pi_m(t))^t$, where $\pi_k(t)$ be money amount invested in kth risky asset at time t, $\mathcal{A}$ denotes the family of all $\mathcal{H}^2$-integrable, $\mathcal{F}_t$-adapted controls.

- Dynamic of controlled wealth process is:

$$dX^\pi(t) = (r(t)X^\pi(t) + \pi(t)^t\alpha(t))dt + \pi(t)^t\sigma(t)dW(t) \tag{1}$$

  with $X_0 = x_0 > 0$. where $\alpha(t) := (\alpha_1(t), \ldots, \alpha_m(t))^t$ and $\alpha_k(t) := \mu_k(t) - r(t)$ for any $k \in \{1, \ldots, m\}$.

- Define

$$\xi(t) := e^{-\int_0^t r(s)ds} e^{-\int_0^t \left(\frac{1}{2}\alpha(s)^t\left(\sigma(s)\sigma(s)^t\right)^{-1}\alpha(s)ds + \alpha(s)^t\left(\sigma(s)^t\right)^{-1}dW(s)\right)}.$$

  $\xi(t)$ is a pricing kernel, i.e., $\xi(t)X^\pi(t)$ is a martingale for any $\pi \in \mathcal{A}$.

- Denote $\xi := \xi(T)$. Then

$$\mathbb{E}[\xi X^\pi(T)] = x_0.$$

Objective Function

- The objective functional is:

$$J(\pi) := \mathbb{E}[U(X^\pi(T))] - \gamma \mathbb{E}[D\left(\mathbb{E}[X^\pi(T)] - X^\pi(T)\right)], \qquad (2)$$

where terminal time $T < \infty$ and $\gamma > 0$ is risk aversion coefficient.

- U is a utility function such that $U : \mathcal{X} \to \mathbb{R}$ is strictly increasing, strictly concave and C^1, here $\mathcal{X}$ is a convex set in $\mathbb{R}$; we extend the definition of U over $\mathbb{R}$ such that $U(x) = -\infty$ for $x \in \mathbb{R}/\mathcal{X}$.

- $D : \mathbb{R} \to \mathbb{R}_+$ is a risk function which measures deviation of the return from the expectation. We assume that D is non-negative, convex and C^1.

- If we set $U(x) = x$ and $D(x) = x^2$, then (2) reduces to classical mean-variance problem. If we set only $U(x) = x$, then (2) reduces to mean-risk problem as in Jin et al. (2005); if we set D to be a convex function with $D(x) = 0$ for $x \le 0$, then (2) is to maximize utility and minimize downside risk of terminal wealth.

Static Optimization

Maximize

$$\mathbb{E}[U(X) - \gamma D(\mathbb{E}[X] - X)]$$

over $X \in \mathcal{L}^2$ subject to

$$\mathbb{E}[\xi X] = x_0.$$

Once one finds the optimal soluiton X^*, one may use the martingale representation theorem to find the replication strategy.

Dual Method

- Assume $D \equiv 0$, no risk function. This is a standard utility maximization.
- Define dual function of U as

$$\tilde{U}(y) = \sup_x (U(x) - xy)$$

for $y \geq 0$. Then

$$U(x) \leq \tilde{U}(y) + xy, \qquad \forall x, y \geq 0.$$

- Assume $Y \in \mathcal{L}^2$. Then

$$\mathbb{E}[U(X)] \leq \mathbb{E}[\tilde{U}(Y)] + \mathbb{E}[XY].$$

- If Y further satisfies

$$\mathbb{E}[XY] \leq C$$

for some constant C and for all feasible X. Then

$$\mathbb{E}[U(X)] \leq \mathbb{E}[\tilde{U}(Y)] + C.$$

- We can find primal-dual relation:

$$\sup_X \mathbb{E}[U(X)] \le \inf_Y \mathbb{E}[\tilde{U}(Y)] + C.$$

- If for some feasible $\hat{X}$ and $\hat{Y}$ we have

$$\mathbb{E}[U(\hat{X})] = \mathbb{E}[\tilde{U}(\hat{Y})] + C, \tag{3}$$

then $\hat{X}$ is optimal for primal problem and $\hat{Y}$ optimal for dual problem. This is because

$$\mathbb{E}[U(\hat{X})] \le \sup_X \mathbb{E}[U(X)] \le \inf_Y \mathbb{E}[\tilde{U}(Y)] + C \le \mathbb{E}[\hat{Y}) + C = \mathbb{E}[U(\hat{X})]$$

- Since

$$\mathbb{E}[U(\hat{X})] \le \mathbb{E}[\tilde{U}(\hat{Y})] + \mathbb{E}[\hat{X}\hat{Y}] \le \mathbb{E}[\tilde{U}(\hat{Y})] + C$$

Equation (3) holds if and only if

$$\mathbb{E}[U(\hat{X})] = \mathbb{E}[\tilde{U}(\hat{Y})] + \mathbb{E}[\hat{X}\hat{Y}] \quad \text{and} \quad \mathbb{E}[\hat{X}\hat{Y}] = C.$$

- The first equality holds if and only if $\hat{X}$ maximizes $x \to U(x) - x\hat{Y}$, or, equivalently, $\hat{X} = (U')^{-1}(\hat{Y})$, which relates optimal solutions of primal and dual problems.
- How can we find C and how can we ensure the second equality?
- Since X must satisfies $\mathbb{E}[\xi X] = x_0$, a natural choice of $\hat{Y}$ would be

$$\hat{Y} = y\xi$$

as this gives $\mathbb{E}[X\hat{Y}] = x_0 y =: C$ for all feasible X.
- We only need to find a y such that

$$\mathbb{E}[\hat{X}\hat{Y}] = y\mathbb{E}[\xi(U')^{-1}(y\xi)] = x_0 y,$$

that is, y is solution to equation

$$\mathbb{E}[\xi(U')^{-1}(y\xi)] = x_0.$$

- If $U'(0) = \infty$ and $U'(\infty) = 0$, then there exists a unique y satisfying the above equation.

Static Optimization (Continue)

- Maximize $\mathbb{E}[U(X) - \gamma D(\mathbb{E}[X] - X)]$ over $X \in \mathcal{L}^2$ subject to $\mathbb{E}[\xi X] = x_0$.
- This is a concave maximization problem. It is natural to try dual method.
- There is a term $\mathbb{E}[X]$ in function D, which is troublesome. For every $\omega \in \Omega$, $X(\omega)$ is basically independent of $\mathbb{E}[X]$, but $\mathbb{E}[X]$ is computed from $X(\omega)$ over all ω, so they are not indepedent.
- We cannot use convex analysis in finite dimensional space, but have to work on convex analysis in infinite dimensional space by defining

$$W(X) = \mathbb{E}[U(X) - \gamma D(\mathbb{E}[X] - X)]$$

for $X \in \mathcal{L}^2$. Define its dual function by

$$\tilde{W}(Y) = \sup_X (W(X) - \langle X, Y \rangle),$$

where $Y \in \mathcal{L}^2$ and $\langle X, Y \rangle$ is inner-product of X and Y, which equals to $\langle X, Y \rangle = \mathbb{E}[XY]$.

- We try to find $\hat{X}, \hat{Y} \in \mathcal{L}^2$ such that

$$\tilde{W}(\hat{Y}) = W(\hat{X}) - \langle \hat{X}, \hat{Y} \rangle,$$

which implies $\hat{X}$ maximizes

$$X \to J(X) := W(X) - \mathbb{E}[X\hat{Y}].$$

- Directional derivative of J at $\hat{X}$ in direction Z is computed by

$$J'(\hat{X}; Z) = \lim_{h \to 0} \frac{J(\hat{X} + hZ) - J(\hat{X})}{h} = \mathbb{E}[ZL(\hat{X}, \hat{Y})],$$

where

$$L(X, Y) = U'(X) - \gamma \mathbb{E}[D'(\mathbb{E}[X] - X)] + \gamma D'(\mathbb{E}[X] - X) - Y.$$

- Since $\hat{X}$ is optimal, $J'(\hat{X}; Z) = 0$ for all Z, which implies

$$L(\hat{X}, \hat{Y}) = 0.$$

- Choose $\hat{Y} = y\xi$ such that $\mathbb{E}[\xi\hat{X}] = x_0$ and $L(\hat{X}, y\xi) = 0$.

Standing Assumptions

Assume terminal wealth $X = X^{\pi}(T)$, $\pi \in \mathcal{A}$, satisfies following conditions:

Condition 1. *$U(X) \in \mathcal{L}^1$ and $D(\mathbb{E}[X] - X) \in \mathcal{L}^1$.*

Condition 2. *There exists $\delta > 0$ such that $D\left(\mathbb{E}\left[X\right] - X - \delta\right) \in \mathcal{L}^1$ and $D\left(\mathbb{E}\left[X\right] - X + \delta\right) \in \mathcal{L}^1$.*

Condition 3. *$X \in \mathcal{L}^2$ and both $U'(X) \in \mathcal{L}^2$ and $D'\left(\mathbb{E}[X] - X\right) \in \mathcal{L}^2$.*

Condition 4. *Initial wealth x_0, lower end point of $\mathcal{X}$, $K := \inf(\mathcal{X}) \in \mathbb{R} \in [-\infty, \infty)$, and pricing kernel $\xi := \xi(T)$ satisfy:*

$$x_0 > \mathbb{E}\left[\xi\right] K.$$

Nonlinear Moment Problem (NMP)

The following theorem provides a NMP as necessary condition of optimality of dynamic control problem (2):

Theorem 5. *Assume the standing assumptions. If* $\hat{\pi}$ *is optimal portfolio of problem (2) and* $X = \hat{X}(T)$ *optimal terminal wealth, then it is necessary for* X *to solve following NMP:*

$$U'(X) + \gamma D'(M - X) = \gamma R + Y\xi, \ \ a.s. \tag{4}$$

where constants $Y, M, R \in \mathbb{R}$ *satisfy*

$$\begin{aligned} \mathbb{E}[\xi X] &= x_0, \\ \mathbb{E}[X] &= M, \\ \mathbb{E}\left[D'(M - X)\right] &= R. \end{aligned}$$

Remark 6. *If* $D \equiv 0$, *then NMP reduces to 2 equations* $Y\xi = U'(X)$ *and* $\mathbb{E}[\xi X] = x_0$ *and can be solved easily.*

Verification Theorem

Theorem 7. *Suppose that there exists $X \in \mathcal{L}^2$ taking values in $\mathcal{X}$ and satisfying Condition 1 and 3 and there exists constants $Y, M, R \in \mathbb{R}$ that solve for NMP (4) almost surely on $\mathcal{X}$, X is the optimal terminal wealth of the portfolio optimization problem (2).*

Proof. Given $X \in \mathcal{L}^2$, by Martingale Representation Theorem, there exists a solution $(\hat{X}(t), \hat{\pi}(t)) \in \mathcal{H}^2 \times \mathcal{H}^2$ satisfying $\hat{X}(T) = X$. For an arbitrary admissible control π, using NMP, we have

$$\frac{d}{d\theta} J(\hat{\pi} + \theta(\pi - \hat{\pi}))\bigg|_{\theta=0} = 0.$$

By e concavity of U and convexity of D, J is concave, and

$$J(\hat{\pi}) \;\geq\; J(\pi) - \frac{J(\hat{\pi} + \theta(\pi - \hat{\pi})) - J(\hat{\pi})}{\theta}.$$

Taking limit leads to $J(\hat{\pi}) \geq J(\pi)$. So $\hat{\pi}$ is optimal control, $X = \hat{X}(T)$ optimal terminal wealth.

Existence of Solution of NMP

We assume $\mathcal{X} = [0, \infty)$ and $U : [0, \infty) \to [0, \infty)$ is strictly concave and C^1, U and D satisfy

$$U'(0) = \infty, U'(\infty) = 0 \text{ and } D'(\infty) = \infty. \tag{5}$$

Thus any utility functions satisfying Inada conditions can be covered. We consider two different types of risk function, namely,

- downside risk function: $D : \mathbb{R} \to [0, \infty)$ is C^1. D is strictly increasing and strictly convex on $(0, \infty)$ and $D(x) = 0$ for $x \leq 0$. The payoff greater than its mean is not penalized, and only the downside risk is considered.
- strictly convex risk function: assume $D : \mathbb{R} \to [0, \infty)$ is strictly convex and C^1 and $D'(-\infty) = -\infty$.

We can show existence of optimal solution of utility-risk portfolio selection problem for these two risk functions.

Key Steps of Proofs

1. For given m, y, there exists $x := I(m, y) \in \mathcal{X}$ satisfying:

$$U'(x) + \gamma D'(m - x) = y,$$

   which would give X a representation

$$X = I\left(M, \gamma R + Y\xi\right).$$

2. For given $Y, M \in (0, \infty)$, there exists a unique $R = R_{Y,M} \in (0, D'(M))$ satisfying

$$\mathbb{E}\left[D'\left(M - X\right)\right] = R \Leftrightarrow \mathbb{E}\left[D'\left(M - I\left(M, \gamma R + Y\xi\right)\right)\right] = R.$$

3. For given $Y \in (0, \infty)$, there exists a unique $M = M_Y \in (0, \infty)$ such that

$$\mathbb{E}[X] = M \Leftrightarrow \mathbb{E}[I\left(M, \gamma R_{Y,M} + Y\xi\right)] = M.$$

4. There exists a (not necessarily unique) $Y^* \in (0, \infty)$ such that

$$\mathbb{E}[\xi X] = x_0 \Leftrightarrow \mathbb{E}[\xi I\left(M_Y, \gamma R_{Y,M_Y} + Y\xi\right)] = x_0.$$

Mean Risk Problem

Assume $U(x) = x$, and set $\mathcal{X} = \mathbb{R}$, Problem (2) reduces to a mean-risk optimization problem:

$$\max_{\pi \in \mathcal{A}} J(\pi) := \mathbb{E}[X^{\pi}(T)] - \gamma \mathbb{E}[D\left(\mathbb{E}[X^{\pi}(T)] - X^{\pi}(T)\right)], \tag{6}$$

As the Inada conditions in (5) does not hold in this case, method developed for utility-risk cannot be directly translated here.

NMP (4) corresponding to (6) can be simplified as follows:

$$1 + \gamma D'\left(M - X\right) = \gamma R + Y\xi, \tag{7}$$

where the numbers $Y, M, R \in \mathbb{R}$ satisfy

$$\begin{aligned} \mathbb{E}[\xi X] &= x_0, \\ \mathbb{E}[X] &= M, \\ \mathbb{E}\left[D'\left(M - X\right)\right] &= R. \end{aligned}$$

Theorem 8. *Suppose that there is a well-defined inverse function for the first derivative of risk function, $I_2 = (D')^{-1}$. If there exists a unique $R \in \mathbb{R}$ so that:*

$$I_2\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right) \in \mathcal{L}^2 \qquad \textit{and} \tag{8}$$

$$\mathbb{E}\left[I_2\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right)\right] = 0, \tag{9}$$

then by setting

$$\begin{aligned} X &:= M - I_2\left(R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma}\right), \\ M &:= \frac{x_0}{\mathbb{E}[\xi]} + \frac{\mathbb{E}\left[\xi I_2\left(R + \frac{\xi Y}{\gamma} - \frac{1}{\gamma}\right)\right]}{\mathbb{E}[\xi]}, \\ Y &:= \frac{1}{\mathbb{E}[\xi]}, \end{aligned} \tag{10}$$

together with R, they solve the reduced NMP (7).

Mean-Variance Case

We further set $D(x) = \frac{1}{2}x^2$. We have $D'(x) = I_2(x) = x$. We have $R = 0$, and also in light of (10), $M = \frac{x_0}{\mathbb{E}[\xi]} + \frac{1}{\gamma}\left(\frac{\mathbb{E}[\xi^2]}{(\mathbb{E}[\xi])^2} - 1\right)$, one can derive the optimal terminal wealth

$$X = \frac{x_0}{\mathbb{E}[\xi]} + \frac{1}{\gamma}\left(\frac{\mathbb{E}[\xi^2]}{(\mathbb{E}[\xi])^2} - \frac{\xi}{\mathbb{E}[\xi]}\right). \tag{11}$$

Optimal trading strategy can be obtained with Martingale Representation Theorem since market is complete. If all market coefficients are deterministic, we can obtain the explicit form of optimal control. We have:

$$\mathbb{E}[\xi] \;=\; e^{-\int_0^T r(s)ds}, \quad \mathbb{E}[\xi^2] = e^{\int_0^T (-2r(s)+\alpha(s)^t\left(\sigma(s)\sigma(s)^t\right)^{-1}\alpha(s))ds}.$$

Using (11), we have

$$X = K - \frac{1}{\gamma}\exp\left[\int_0^T r(s)ds\right]\xi,$$

where $K := \exp\left[\int_0^T r(s)ds\right] x_0 + \frac{1}{\gamma}\exp\left[\int_0^T \alpha(s)^t\left(\sigma(s)\sigma(s)^t\right)^{-1}\alpha(s)ds\right]$. Since

$\{\hat{X}(t)\xi(t)\}$ is a martingale, we have

$$\begin{aligned}\hat{X}(t)\xi(t) &= K\mathbb{E}[\xi|\mathcal{F}_t] - \frac{1}{\gamma}e^{\int_0^T r(s)ds}\mathbb{E}[\xi^2|\mathcal{F}_t] \\ &= K\xi(t)e^{-\int_t^T r(s)ds} - \frac{1}{\gamma}e^{\int_0^T r(s)ds}\left(\xi(t)\right)^2 e^{\int_t^T (-2r(s)+\alpha(s)^t\left(\sigma(s)\sigma(s)^t\right)^{-1}\alpha(s))ds} \qquad (12)\end{aligned}$$

Applying Itô's formula to (12), we have

$$d\left(\hat{X}(t)\xi(t)\right) = K\xi(t)e^{\int_t^T -r(s)ds}\alpha(t)^t\left(\sigma(t)^t\right)^{-1} - 2\hat{X}(t)\xi(t)\alpha(t)^t\left(\sigma(t)^t\right)^{-1}dW(t). \qquad (13)$$

On the other hand, by applying Itô's formula to $\hat{X}(t)\xi(t)$ directly, we have

$$d\left(\hat{X}(t)\xi(t)\right) = \xi(t)\left(-\alpha(t)^t\left(\sigma(t)^t\right)^{-1}\hat{X}(t) + \pi(t)^t\sigma(t)\right)dW(t), \quad (14)$$

Comparing coefficients of (13) and (14) we have optimal control

$$\hat{\pi}(t) = \left(\sigma(t)\sigma(t)^t\right)^{-1}\alpha(t)\left(-\hat{X}(t) + Ke^{\int_t^T -r(s)ds}\right).$$

Mean-Weighted Variance Case

Consider risk function

$$D(x) = ax_+^2 + bx_-^2,$$

where $a \geq b > 0$. Same problem is studied in Jin et al (2005), we apply NMP in Theorem 5 to obtain same solution.

$$D'(x) = 2ax_+ - 2bx_-, \qquad I_2(x) = \frac{1}{2a}x_+ - \frac{1}{2b}x_-.$$

By intermediate value theorem, there exists a unique $R \in (0, \frac{1}{\gamma})$ such that (9) and (8) are satisfied. Solution of mean weighted variance problem is:

$$\begin{aligned} X \;=\; & \frac{1}{\mathbb{E}[\xi]}\left(x_0 + \mathbb{E}\left[-\frac{\xi\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right)_+}{2a} + \frac{\xi\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right)_-}{2b}\right]\right) \\ & -\frac{\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right)_+}{2a} + \frac{\left(R + \frac{\xi}{\gamma\mathbb{E}[\xi]} - \frac{1}{\gamma}\right)_-}{2b}, \end{aligned} \tag{15}$$

where R is the unique root of the equation

$$\mathbb{E}\left[\frac{\left(R+\frac{\xi}{\gamma\mathbb{E}[\xi]}-\frac{1}{\gamma}\right)_+}{2a}-\frac{\left(R+\frac{\xi}{\gamma\mathbb{E}[\xi]}-\frac{1}{\gamma}\right)_-}{2b}\right]=0. \tag{16}$$

If $a=b=1$, then mean-weighted-variance model becomes mean-variance. We get $R=0$ from (16) and get the same solution as in (11) from (15). Optimal feedback control of mean-weighted-variance problem is

$$\begin{aligned}\hat{\pi}(t) \;=\; & \frac{\alpha}{\sigma^2}(-X(t)+\left(K_1+\frac{1-\gamma R}{2b\gamma}\right)e^{-r(T-t)} \\ & +\frac{1-\gamma R}{2\gamma}\left(\frac{1}{a}-\frac{1}{b}\right)e^{-r(T-t)}\Phi\left(Z_t+\frac{\alpha}{\sigma}\sqrt{T-t}\right)),\end{aligned}$$

where Φ is cdf of standard normal random variable and,

$$Z_t=-\frac{\sigma}{\alpha}\frac{\ln(1-\gamma R)+\frac{\alpha^2}{2\sigma^2}T+\frac{\alpha}{\sigma}W(t)}{\sqrt{T-t}}$$

Note that current value of BM $W(t)$ can be easily computable by the current stock price $S(t)$.

Mean-Semivariance Case

Assume $D(x) = \frac{1}{2}x_+^2$. Then $D'(x) = x_+$. We revisit the negative result first obtained in Jin et al. (2005) via our approach.

Theorem 9. *There is no optimal solution for mean-semivariance problem.*

Proof. Assume the contrary that there exists an admissible optimal control $\hat{\pi}$ and $\hat{X}$ is the corresponding optimal wealth process.

$$D'\left(\mathbb{E}\left[\hat{X}(T)\right] - \hat{X}(T)\right) = \left(\mathbb{E}\left[\hat{X}(T)\right] - \hat{X}(T)\right)_+ \leq \left|\mathbb{E}\left[\hat{X}(T)\right]\right| + \left|\hat{X}(T)\right|,$$

thus, $\hat{X}(T)$ satisfies Condition 3. Also,

$$D\left(\mathbb{E}\left[\hat{X}(T)\right] - \hat{X}(T) + k\right) \leq \left(\mathbb{E}\left[\hat{X}(T)\right]\right)^2 + \left(\hat{X}(T)\right)^2 + k^2,$$

for any constant $k \in \mathbb{R}$, thus $\hat{X}(T)$ fulfills Condition 2. Hence, by Theorem

5, it is necessary for $X = \hat{X}(T)$ to solve the following NMP:

$$Y\xi + \gamma R - 1 = \gamma (M - X)_+, \text{ a.s.} \tag{17}$$

where numbers $Y, M, R \in \mathbb{R}$ satisfy

$$\begin{aligned} \mathbb{E}[\xi X] &= x_0, \\ \mathbb{E}[X] &= M, \\ \mathbb{E}\left[(M - X)_+\right] &= R. \end{aligned}$$

Firstly, by taking expectation on both sides of (17), we have $Y = \frac{1}{\mathbb{E}[\xi]} > 0$. If $\gamma R - 1 \geq 0$, then by (17), $\gamma (M - X)_+ > 0$ almost surely, which contradicts to $\mathbb{E}[X] = M$. If $\gamma R - 1 < 0$, there exists some $\xi_0 > 0$ such that $\gamma R - 1 + Y\xi < 0$ for all $\xi \in (0, \xi_0)$, which contradicts to (17). Thus, NMP has no solution, we can conclude that mean-downside-risk problem does not admit an optimal solution.

□

Conclusion

- We provide a comprehensive study on utility-risk portfolio selection.
- We show the optimal terminal wealth must satisfy a mean field equation by considering first order optimality condition, which leads to a primitive static problem, called NMP, that has never appeared in the literature.
- We use NMP to show existence of optimal solution for a variety of utility-risk problems, incluidng utility-downside-risk and utility-strictly-convex-risk problems.
- We establish sufficient condition for existence of solution of mean-risk problem, including power, variance, weighted-variance, and exponential risk functions.