The Geometry of Model Uncertainty

Mathias Beiglböck

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1) background – model uncertainty and optimal transport

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2) particular aspect: Skorokhod embedding

Model Uncertainty – Overview

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basic object: **stock price** (random process)

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all call prices known $\iff$ $S_t \sim_\mathbb{P} \mu_t$

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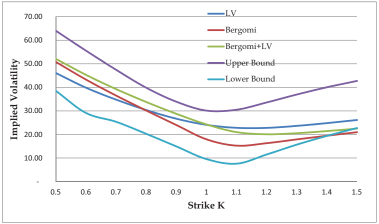

lower/upper prices versus (local) Bergomi and LV models

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B., Henry-Labordere, Penkner / Galichon, Touzi ('13) $\longrightarrow$ *transport approach*

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$$S_t \sim_{\mathbb{P}} \mu_t$$

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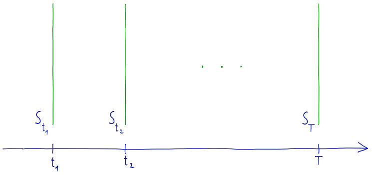

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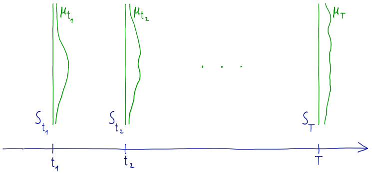

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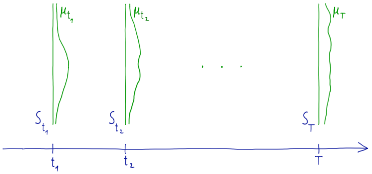

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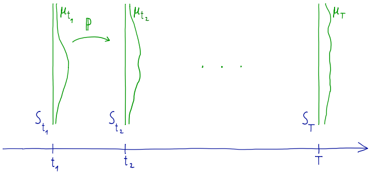

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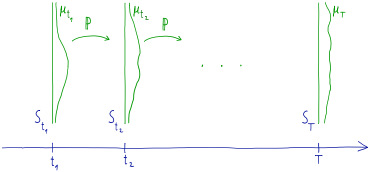

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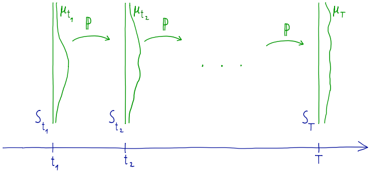

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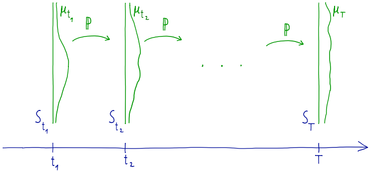

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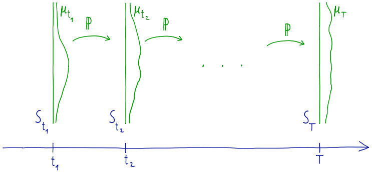

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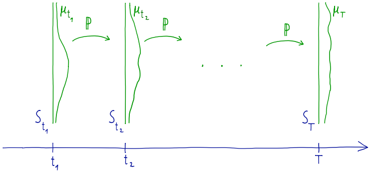

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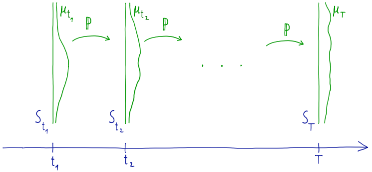

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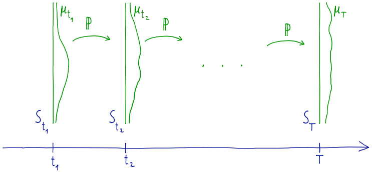

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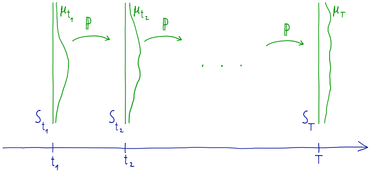

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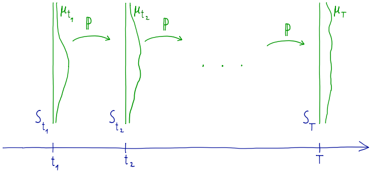

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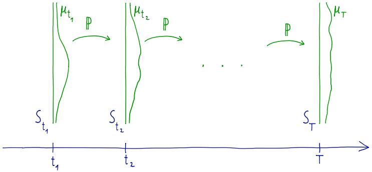

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geometric description of extreme models

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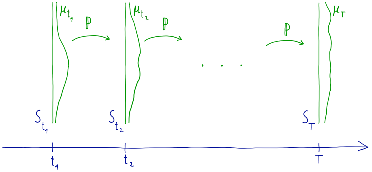

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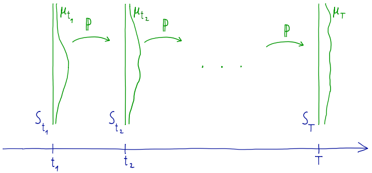

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cont. time – *geometry* of Skorokhod embedding

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- Skorokhod '61
- Root '69
- Rost '71
- Azema-Yor '79
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Skorokhod embedding problem (SEP)

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recent surveys:

- Obloj '04
- Hobson '11

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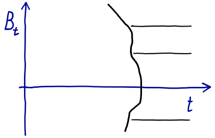

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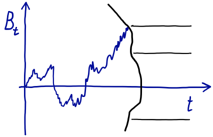

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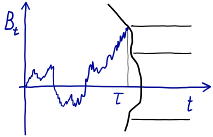

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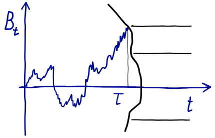

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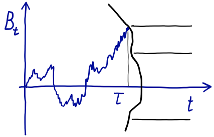

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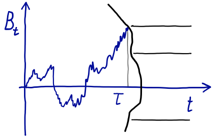

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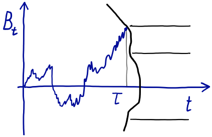

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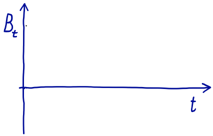

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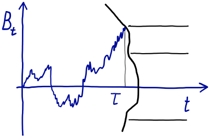

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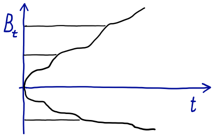

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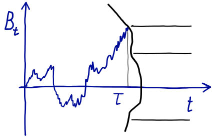

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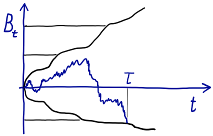

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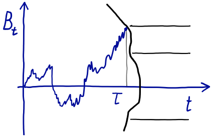

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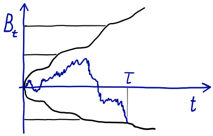

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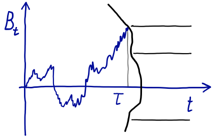

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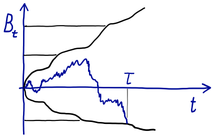

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Azema-Yor

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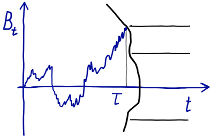

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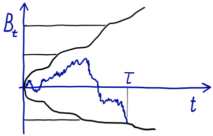

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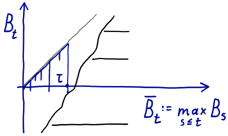

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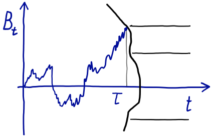

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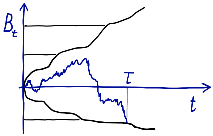

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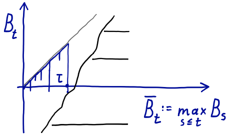

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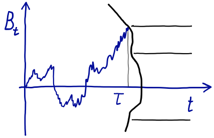

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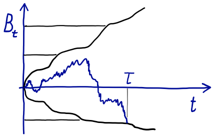

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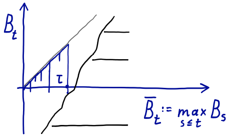

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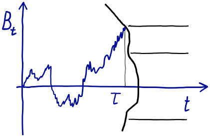

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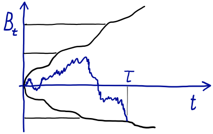

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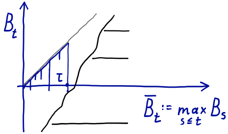

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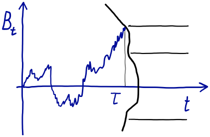

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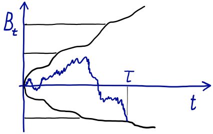

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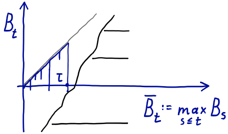

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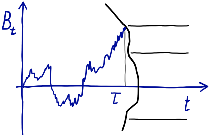

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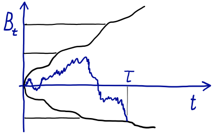

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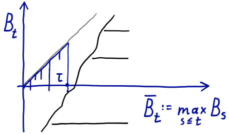

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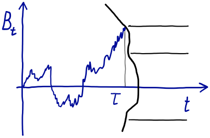

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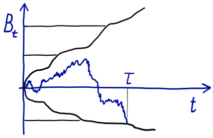

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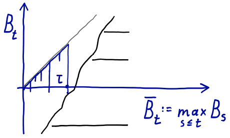

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optimal embeddings: basis for *all* known extremal models

Transport Approach to Skorokhod Embedding [BCH15]

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Thm – Existence

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$\exists$ *monotone* Γ s.t. $\mathbb{P}((B_s)_{s\le\tau}) \in \Gamma) = 1$

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Φ upper semi-continuous $\implies$ dual theory of Skorokhod embededding

Thm – Monotonicity Principle

Φ Borel, τ optimal $\implies$ $\underbrace{\textit{support of } \tau \textit{ is pathwise optimal}}_{\exists \textit{ monotone } \Gamma \text{ s.t. } \mathbb{P}((B_s)_{s \le \tau}) \in \Gamma) = 1}$

Γ *monotone* $:\iff$ Γ cannot be improved by pathwise modifications

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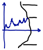
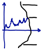
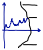
Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$

Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \to \min$

Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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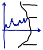
Thm [Root69]: given $\mu \Rightarrow\ \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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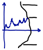
Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu$, $\ \mathbb{E}\tau_R^2 \to \min$

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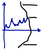
Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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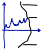
Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \to \min$

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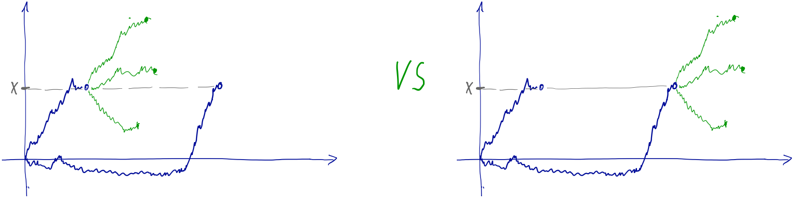
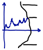


Thm [Root69]: given $\mu \Rightarrow \;\; \exists \tau_R$ of  - type, $\;\; B_{\tau_R} \sim \mu, \;\; \mathbb{E}\tau_R^2 \to \min$

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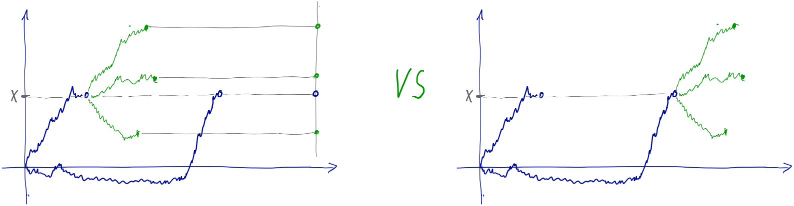
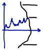


Thm [Root69]: given $\mu \Rightarrow$ $\exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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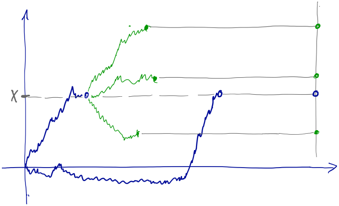

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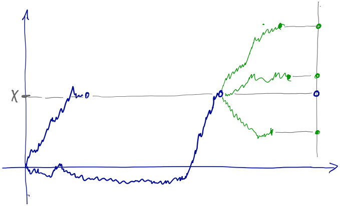
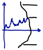


Thm [Root69]: given $\mu \Rightarrow \;\exists \tau_R$ of  - type, $\;B_{\tau_R} \sim \mu$, $\;\mathbb{E}\tau_R^2 \to$ min

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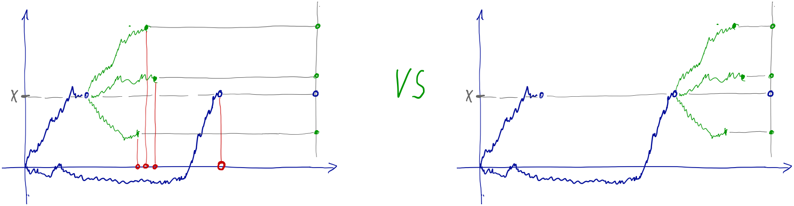
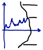


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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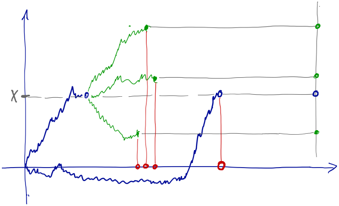

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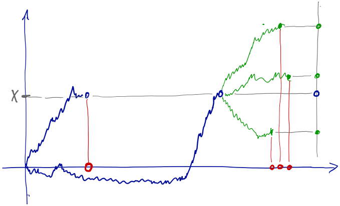
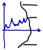


Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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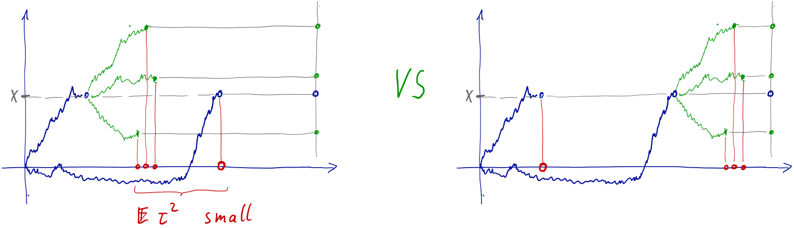
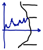


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of  - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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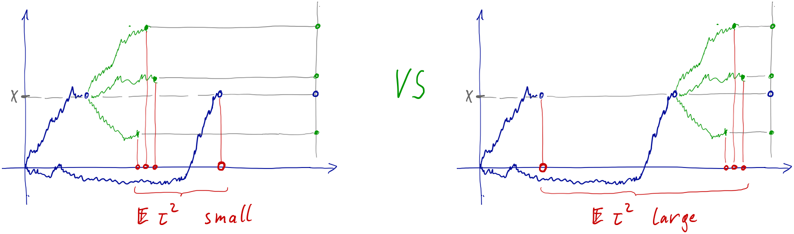
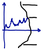


Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \to \min$

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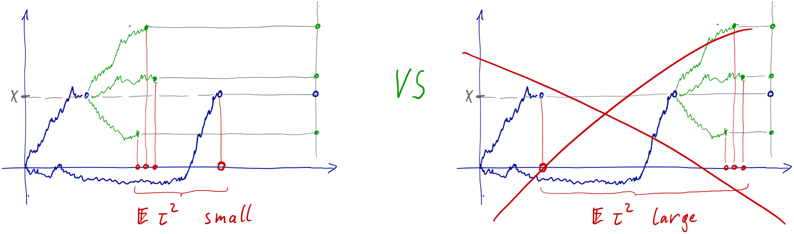
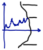


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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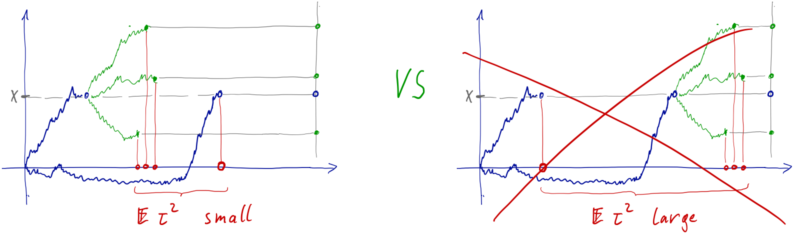

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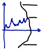
Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \to \min$

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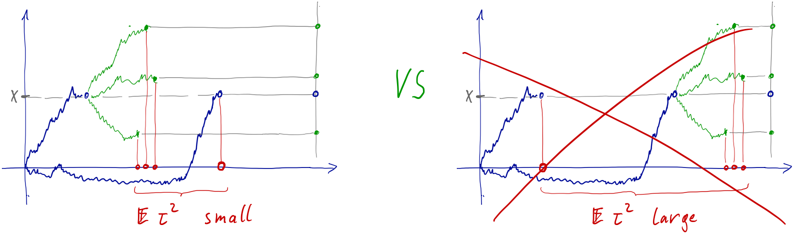

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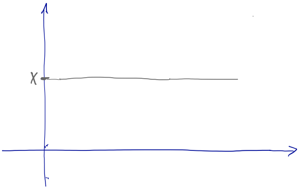
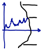


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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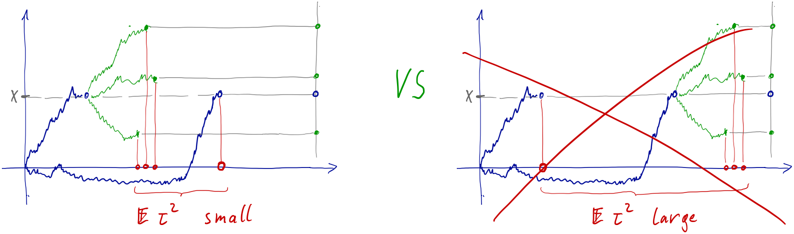

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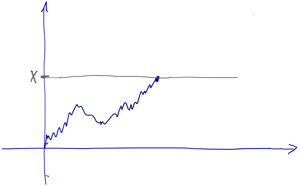
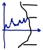


Thm [Root69]: given $\mu \Rightarrow$ $\exists \tau_R$ of  - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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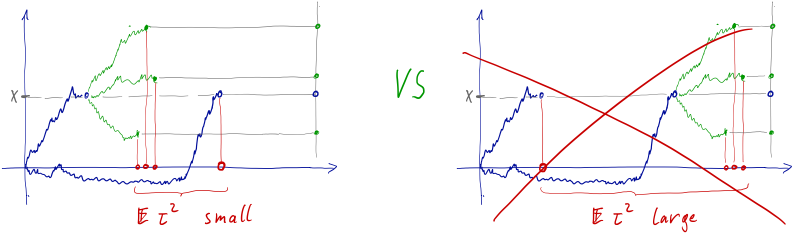

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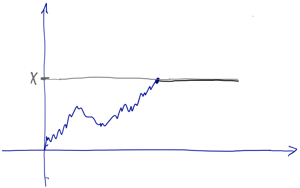
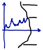


Thm [Root69]: given $\mu \Rightarrow$ $\exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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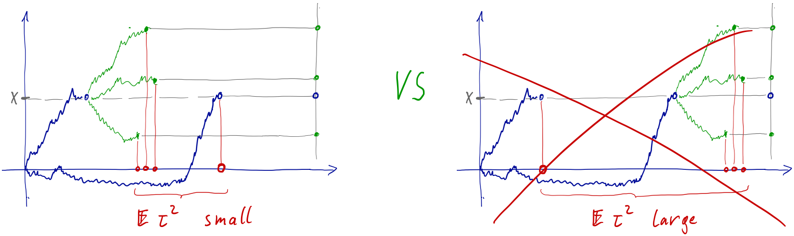

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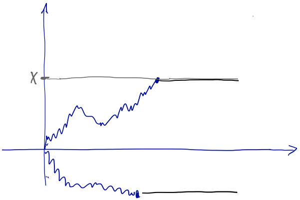
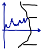


Thm [Root69]: given $\mu \Rightarrow$ $\exists \tau_R$ of - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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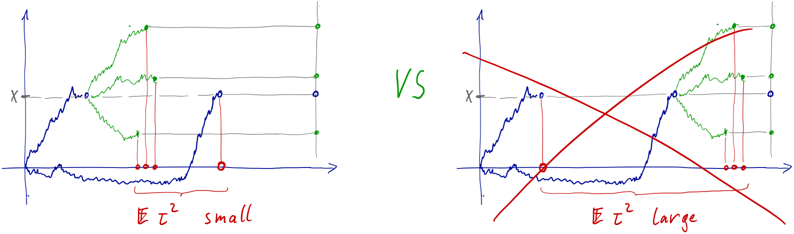

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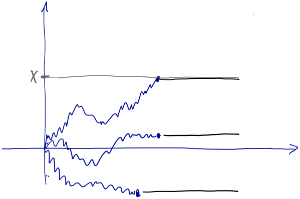
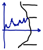


Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \rightarrow$ min

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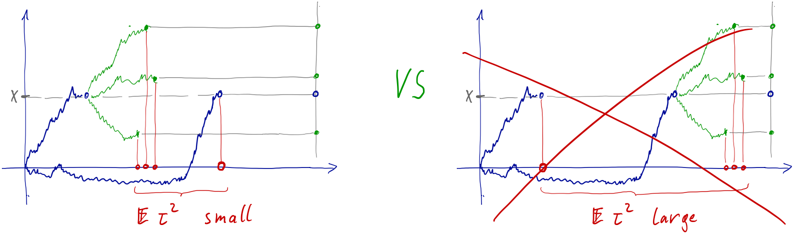

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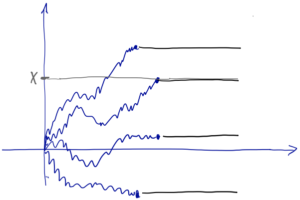
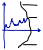


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of  - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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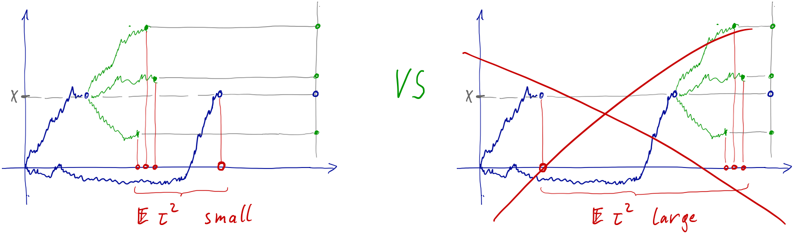

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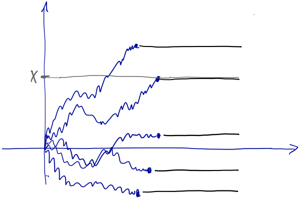
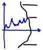


Thm [Root69]: given $\mu \Rightarrow \exists \tau_R$ of  - type, $B_{\tau_R} \sim \mu$, $\mathbb{E}\tau_R^2 \to \min$

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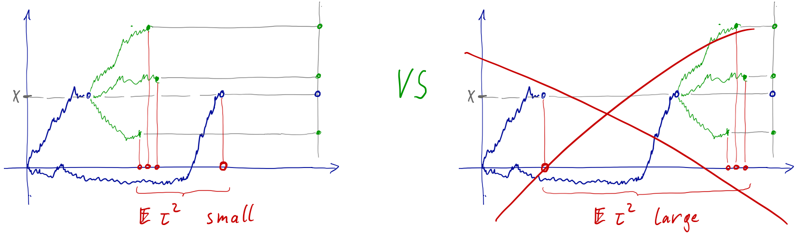

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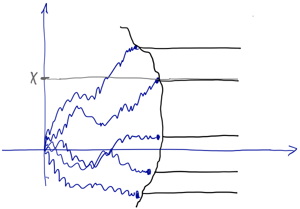
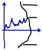


Thm [Root69]: given $\mu \Rightarrow \ \exists \tau_R$ of - type, $\ B_{\tau_R} \sim \mu, \ \ \mathbb{E}\tau_R^2 \to \min$

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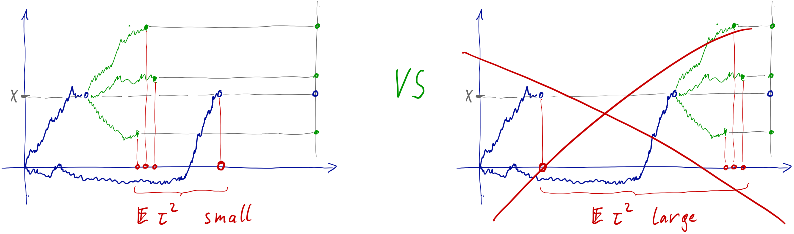

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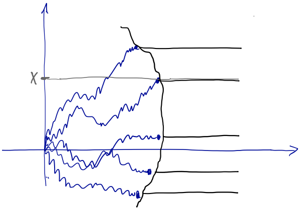

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transport approach [BCH16]:

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transport approach [BCH16]: *all* optimal emb. extend in *full* generality

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