Stochastic algorithms for the approximative pricing of financial derivatives

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Joint works with

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Introduction

Consider $T \in (0, \infty)$, $d \in \mathbb{N}$ and sufficiently regular functions $g\colon \mathbb{R}^d \to \mathbb{R}$, $f\colon [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\mu\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $u\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that $u(T, x) = g(x)$ and

$$(\tfrac{\partial}{\partial t} u)(t, x) + f\big(t, x, u(t, x), \sigma(t, x)(\nabla_x u)(t, x)\big) + \langle \mu(t, x), (\nabla_x u)(t, x) \rangle \\ + \tfrac{1}{2} \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma(t, x)\sigma(t, x)^* (\operatorname{Hess}_x u)(t, x)\big) = 0$$

for $(t, x) \in [0, T) \times \mathbb{R}^d$.

- **Black-Scholes model** Consider $T, \beta > 0$, $\alpha \in \mathbb{R}$ and

$$\frac{\partial}{\partial t} X_t = \alpha\, X_t + \beta\, X_t\, \frac{\partial}{\partial t} dW_t$$

  for $t \in [0, T]$, where $(W_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion.

- **Heston model** Consider $\alpha, \gamma \in \mathbb{R}$, $\beta, \delta, X_0^{(1)}, X_0^{(2)} > 0$, $\rho \in [-1, 1]$ and

$$\frac{\partial}{\partial t} X_t^{(1)} = \alpha\, X_t^{(1)} + \sqrt{X_t^{(2)}}\, X_t^{(1)}\, \frac{\partial}{\partial t} W_t^{(1)}$$

$$\frac{\partial}{\partial t} X_t^{(2)} = \delta - \gamma X_t^{(2)} + \beta \sqrt{X_t^{(2)}} \left(\rho\, \frac{\partial}{\partial t} W_t^{(1)} + \sqrt{1-\rho^2}\, \frac{\partial}{\partial t} W_t^{(2)}\right)$$

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Theorem (Hairer, Hutzenthaler & J 2015 AOP)

Let $T \in (0,\infty)$, $d \in \{4,5,\dots\}$, $\xi \in \mathbb{R}^d$. *Then there exist globally bounded* $\mu, \sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *every Brownian motion* $W\colon [0,T] \times \Omega \to \mathbb{R}$, *every solution* $X\colon [0,T]\times\Omega \to \mathbb{R}^d$ *of*

$$dX_t = \mu(X_t) + \sigma(X_t)\,dW_t, \qquad t \in [0,T], \qquad X_0 = \xi,$$

and every $Y^N\colon \{0,1,\dots,N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, *with* $\forall\, N \in \mathbb{N}, n \in \{0,1,\dots,N-1\}\colon\ Y_0^N = X_0$ *and*

$$Y_{n+1}^N = Y_n^N + \mu(Y_n^N)\tfrac{T}{N} + \sigma(Y_n^N)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)$$

(Euler-Maruyama approximations) we have $\forall\, \alpha \in [0,\infty)$:

$$\lim_{N\to\infty}\left(N^\alpha \left\| \mathbb{E}\big[X_T\big] - \mathbb{E}\big[Y_N^N\big]\right\|\right) = \begin{cases} 0 & : \alpha = 0 \\ \infty & : \alpha > 0 \end{cases}.$$

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Let $T \in (0, \infty)$, $d \in \{4, 5, \dots\}$, $\xi \in \mathbb{R}^d$. *Then there exist globally bounded* $\mu, \sigma \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *every Brownian motion* $W\colon [0, T] \times \Omega \to \mathbb{R}$, *every solution* $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ *of*

$$dX_t = \mu(X_t) + \sigma(X_t)\, dW_t, \qquad t \in [0, T], \qquad X_0 = \xi,$$

and every $Y^N\colon \{0, 1, \dots, N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, *with* $\forall\, N \in \mathbb{N}, n \in \{0, 1, \dots, N-1\}\colon\ Y_0^N = X_0$ *and*

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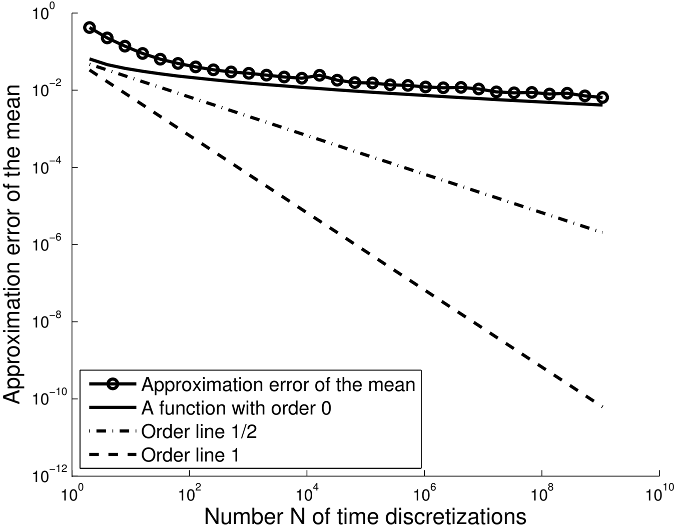
Plot of $\left\|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right]\right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



Theorem (Gerencsér, J, & Salimova 2016)

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Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}$ be a Brownian motion, let $X \colon [0, T] \times \Omega \to \mathbb{R}$ be a solution of

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Theorem (Hefter & J 2016)

Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W\colon [0, T] \times \Omega \to \mathbb{R}$ *be a Brownian motion, let* $X\colon [0, T] \times \Omega \to \mathbb{R}$ *be a solution of*

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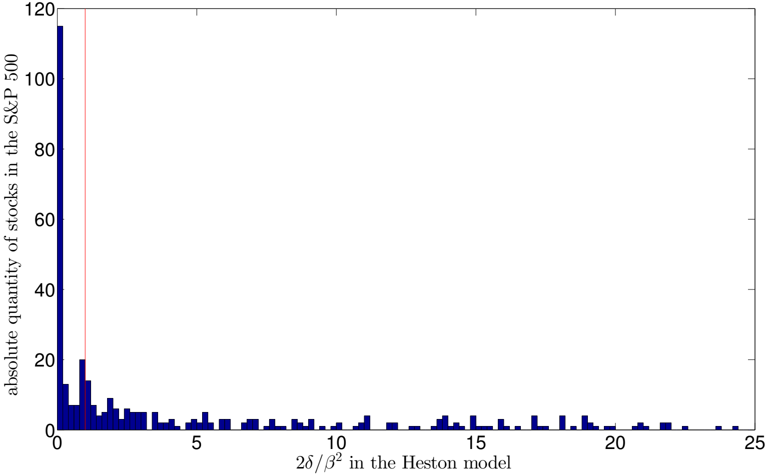
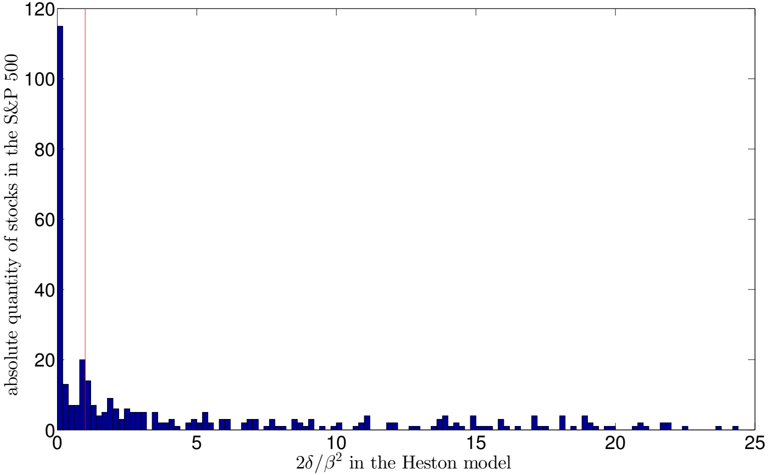


The **S&P 500** (the Standard & Poor's 500) is a stock market index.

In Hutzenthaler, J & Noll 2016 we calibrate 498 stocks from the S&P 500 within the **Heston model**: 359 stocks satisfy $\frac{2\delta}{\beta^2} \leq 25$, 162 stocks ($\approx 32\%$) satisfy $\frac{2\delta}{\beta^2} < 1$. More than 100 stocks ($= 20\%$) satisfy $\frac{2\delta}{\beta^2} \leq \frac{1}{10}$.



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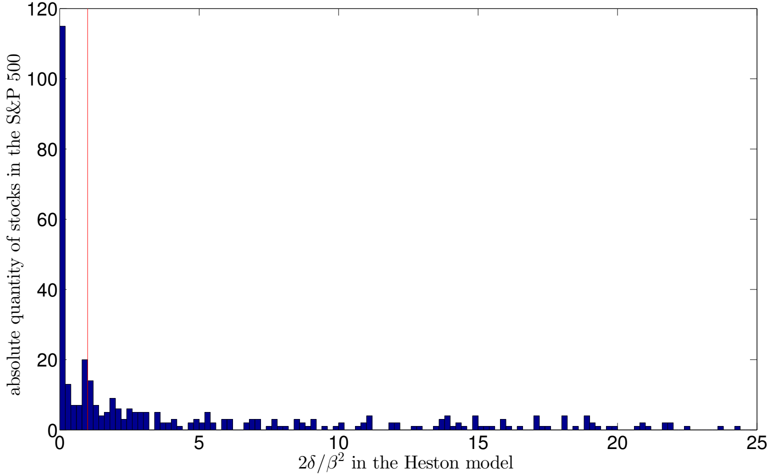

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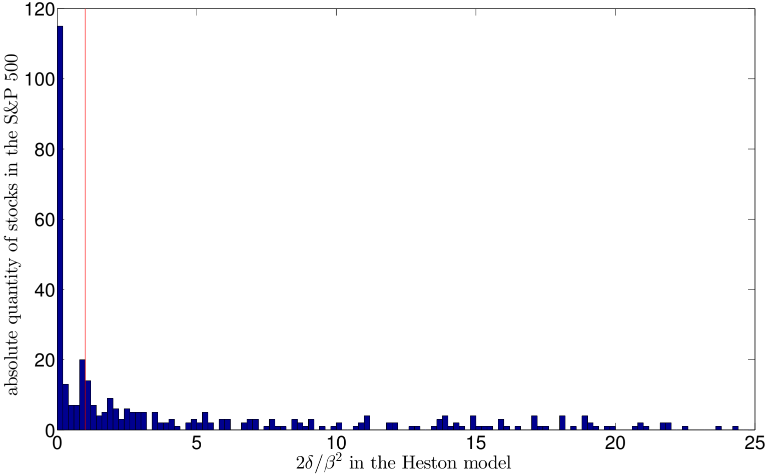

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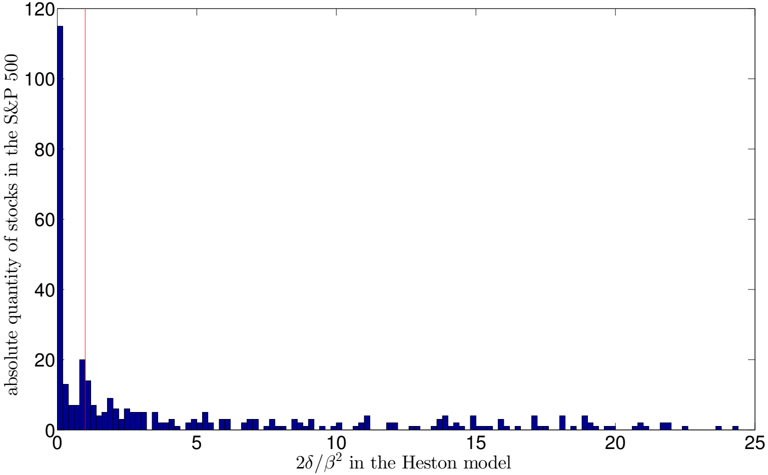

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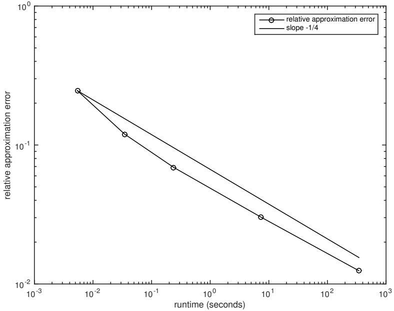
Let $\Theta = \cup_{n\in\mathbb{N}}\mathbb{R}^n$, let $(q_s^{k,l,\rho})_{k,l\in\mathbb{N}_0,\rho\in(0,\infty),s\in[0,T)} \subseteq \mathcal{Q}_T$, $(m^g_{k,l,\rho})_{k,l\in\mathbb{N}_0,\rho\in(0,\infty)}, (m^f_{k,l,\rho})_{k,l\in\mathbb{N}_0,\rho\in(0,\infty)} \subseteq \mathbb{N}$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W^\theta\colon[0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, be independent standard $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motions, for every $l\in\mathbb{Z}$, $\rho\in(0,\infty)$, $\theta\in\Theta$, $x\in\mathbb{R}^d$, $s\in[0,T)$, $t\in[s,T]$ let $\mathcal{X}^{l,\rho,\theta}_{x,s,t}\colon\Omega\to\mathbb{R}^d$, $\mathcal{D}^{l,\rho,\theta}_{x,s,t}\colon\Omega\to\mathbb{R}^{d\times d}$ and $\mathcal{I}^{l,\rho,\theta}_{x,s,t}\colon\Omega\to\mathbb{R}^{1+d}$ be functions, and for every $\theta\in\Theta$, $\rho\in(0,\infty)$ let $\mathbf{U}^\theta_{k,\rho}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}^{d+1}$, $k\in\mathbb{N}_0$, be functions which satisfy for all $k\in\mathbb{N}$, $(s,x)\in[0,T)\times\mathbb{R}^d$ that

$$
\begin{aligned}
&\mathbf{U}^\theta_{k,\rho}(s,x)\\
&=\sum_{l=0}^{k-1}\sum_{i=1}^{m^g_{k,l,\rho}}\frac{\left[g(\mathcal{X}^{l,\rho,(\theta,l,-i)}_{x,s,T})-\mathbb{1}_{\mathbb{N}}(l)\,g(\mathcal{X}^{l-1,\rho,(\theta,l,-i)}_{x,s,T})-\mathbb{1}_{\{0\}}(l)\,g(x)\right]}{m^g_{k,l,\rho}}\,\mathcal{I}^{l,\rho,(\theta,l,-i)}_{x,s,T}\\
&+(g(x),0)+\sum_{l=0}^{k-1}\sum_{i=1}^{m^f_{k,l,\rho}}\sum_{t\in[s,T]}\frac{q_s^{k,l,\rho}(t)}{m^f_{k,l,\rho}}\left[f\Big(t,\mathcal{X}^{\rho,k-l,(\theta,l,i)}_{x,s,t},\mathbf{U}^{(\theta,l,i,t)}_{l,\rho}\big(t,\mathcal{X}^{k-l,\rho,(\theta,l,i)}_{x,s,t}\big)\Big)\right.\\
&\left.-\mathbb{1}_{\mathbb{N}}(l)\,f\Big(t,\mathcal{X}^{k-l,\rho,(\theta,l,i)}_{x,s,t},\mathbf{U}^{(\theta,-l,i,t)}_{[l-1]^+,\rho}\big(t,\mathcal{X}^{k-l,\rho,(\theta,l,i)}_{x,s,t}\big)\Big)\right]\mathcal{I}^{k-l,\rho,(\theta,l,i)}_{x,s,t}.
\end{aligned}
$$

Allen-Cahn equation



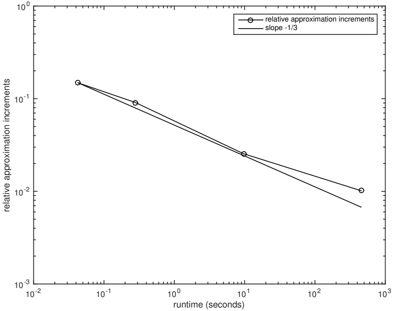


Figure: Relative approximation errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10}|\mathbf{U}^{i,[1]}_{\rho,\rho}(0,x_0)-\mathrm{v}|$ for $\rho\in\{1,2,\ldots,5\}$ against the average runtime in the case $d=1$ ($u(0,x_0)\approx\mathrm{v}=0.905$). Right: Relative approximation increments $\left(\frac{1}{10}\sum_{i=1}^{10}|\mathbf{U}^{i,[1]}_{\rho+1,\rho+1}(0,x_0)-\mathbf{U}^{i,[1]}_{\rho,\rho}(0,x_0)|\right)\Big/\left(\frac{1}{10}|\sum_{i=1}^{10}\mathbf{U}^{i,[1]}_{7,7}(0,x_0)|\right)$ for $\rho\in\{1,2,3,4\}$ against the average runtime in the case $d=100$ ($u(0,x_0)\approx 0.317$).

Numerical simulations in MATLAB with an Intel i7 CPU with 2.8 GHz Intel and 16 GB RAM.

Pricing with different interest rates for borrowing and lending

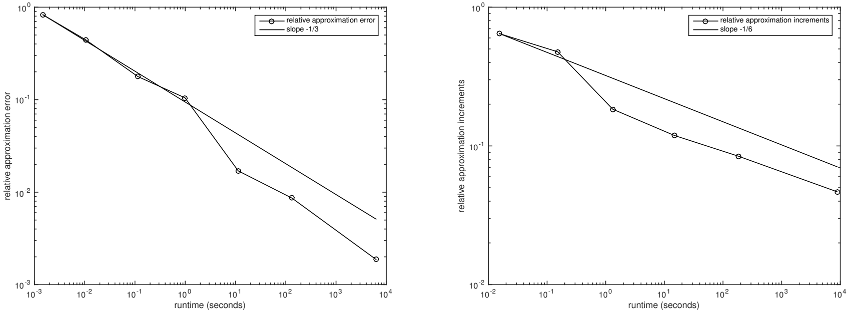


Figure: Relative approximation errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10}|\mathbf{U}^{i,[1]}_{\rho,\rho}(0,x_0)-\mathrm{v}|$ for $\rho\in\{1,2,\ldots,7\}$ against the average runtime in the case $d=1$ ($u(0,x_0)\approx\mathrm{v}=7.156$). Right: Relative approximation increments $\left(\frac{1}{10}\sum_{i=1}^{10}|\mathbf{U}^{i,[1]}_{\rho+1,\rho+1}(0,x_0)-\mathbf{U}^{i,[1]}_{\rho,\rho}(0,x_0)|\right)\Big/\left(\frac{1}{10}|\sum_{i=1}^{10}\mathbf{U}^{i,[1]}_{7,7}(0,x_0)|\right)$ for $\rho\in\{1,2,\ldots,6\}$ against the average runtime in the case $d=100$ ($u(0,x_0)\approx 21.299$).

Runtime needed to compute one realization of $\mathbf{U}^1_{6,6}(0, x_0)$ against dimension $d \in \{5, 6, \ldots, 100\}$ for the **pricing with different interest rates** example.