

Semiparametric Bayesian estimation in copula models

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Seminar at WU Institute for Statistics and Mathematics

Vienna

16 November 2018

Bayesian inference is based on the posterior distribution:

$$\pi(\theta; \mathbf{y}) \propto \pi(\theta) L(\theta; \mathbf{y})$$

- Posterior distribution has a closed-form expression just in a few and simple cases \Rightarrow computational methods

ABC Rejection algorithm

For $l = 1, \dots, N$ **do**

Repeat

 Generate θ' from the prior distribution $\pi(\cdot)$

 Generate z from the likelihood $f(\cdot | \theta')$

Until $\rho(\eta(z), \eta(y)) < \varepsilon$

 Set $\theta_l = \theta'$

End For

- Bayesian synthetic likelihood [Drovandi et al., 2015, Price et al., 2016]

$$\pi_n(\theta|s_{obs}) \propto \mathcal{N}(s_{obs}; \mu_n(\theta), \Sigma_n(\theta))\pi(\theta)$$

- Bayesian empirical likelihood [Mengersen et al., 2013]

$$\pi(\theta|y) \propto L_{EL}(\theta)\pi(\theta)$$

- Bayesian bootstrap likelihood [Zhu et al., 2015]

$$\pi(\theta|y) \propto L_{BL}(\theta)\pi(\theta)$$

Empirical Likelihood

Empirical likelihood is a way of producing a nonparametric likelihood for a quantity of interest [Owen, 2001]. Schennach [2005] proposes a **Bayesian exponentially tilted empirical likelihood**.

Consider a given set of generalized **moment conditions**

$$E_F(h(X, \varphi)) = 0,$$

where $h(\cdot)$ is a known function, and φ is the quantity of interest.

$L_{BEL}(\varphi; \mathbf{x})$ is defined as the system of weights (p_1, \dots, p_n) obtained as solution of

$$\max_{(p_1, \dots, p_n)} \sum_{i=1}^n (-p_i \log p_i)$$

under constraints

$$0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1 \quad \sum_{i=1}^n h(x_i, \varphi) p_i = 0$$

[Owen's maximisation problem was $\max_{(p_1, \dots, p_n)} \prod_{i=1}^n p_i$]

The Bayesian use of the empirical likelihood I

We are interested in a function φ and in its posterior

$$\pi(\phi|y) \propto \int_N p(y|v, \phi)\pi(v|\phi)\pi(\phi)dv$$

or

$$\pi(\phi|y) \propto \lim_{N \rightarrow \infty} \int_N p(y|\xi_N, \phi)\pi(\xi_N|\phi)\pi(\phi)d\xi_N$$

Then the distribution C can be represented as

$$C = (\varphi, C^*)$$

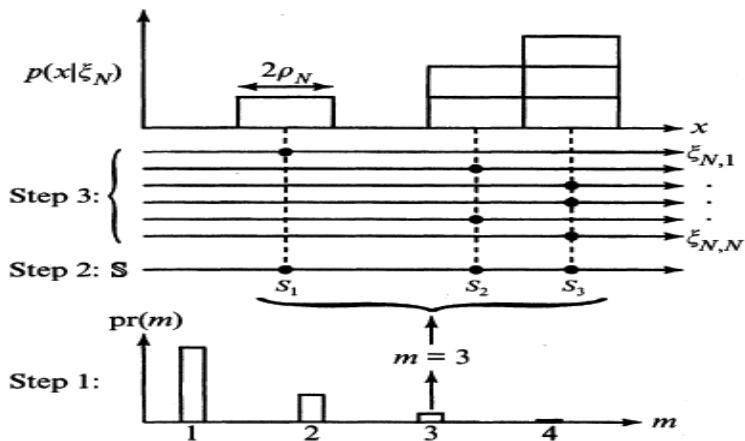
where C^* belongs to an infinite dimensional metric space (H, d_H) .

L_{BEL} may be seen as the derivation of the integrated likelihood for ϕ

$$L_{BEL}^{(\lambda)}(\phi; y) = \int_{\Xi} L(\phi, \xi; y)d\Pi(\xi)$$

where $\Pi(\xi)$ is the prior process implicitly induced by L_{BEL} .

The Bayesian use of the empirical likelihood II



Why copulas?

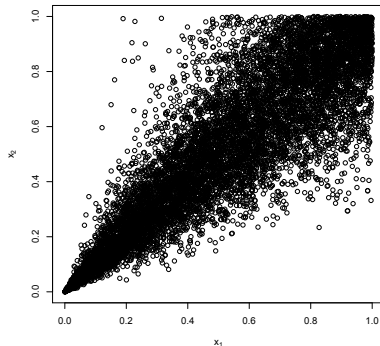


Figure: The Clayton copula exhibits greater dependence in the negative tail

Sklar's Theorem

A copula model is a way of representing the joint distribution of a random vector $\mathbf{X} = (X_1, \dots, X_d)$. Given an d -variate cumulative distribution function (CDF) \mathbf{F} , it is possible to show [Sklar, 1959] that there always exists an d -variate function $C : [0, 1]^d \rightarrow [0, 1]$, such that

$$\mathbf{F}(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

where F_j is the marginal CDF of X_j .

Therefore, in case that the multivariate distribution has a density \mathbf{f} , and this is available, it holds further that

$$\mathbf{f}(x_1, \dots, x_d) = c(F_1(x_1), \dots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d)$$

Example of copula functions

- Clayton copula: $C_\theta(u, v) = [\max\{u^{-\theta} + v^{-\theta} - 1; 0\}]^{-\frac{1}{\theta}}$ for $\theta \in [-1, 1) \setminus \{0\}$
- Gumbel copula: $C_\theta(u, v) = \exp[-((-\log(u))^\theta + (-\log(v))^\theta)^{\frac{1}{\theta}}]$ for $\theta \in [1, \infty]$
- Frank copula $C_\theta(u, v) = -\frac{1}{\theta} \log \left[1 + \frac{(\exp(-\theta u) - 1)(\exp(-\theta v) - 1)}{\exp(-\theta) - 1} \right]$ for $\theta \in \mathbb{R} \setminus 0$

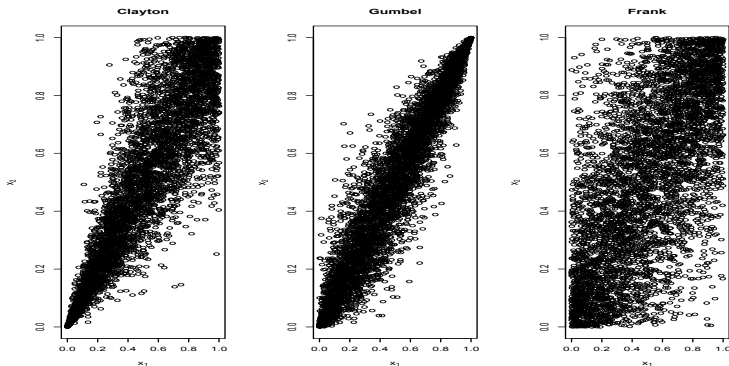


Figure: Simulations from different copula functions

- **Frequentist methods:**

- *Inference from the margins* [Joe, 2015]
- method of moments [Oh and Patton, 2013]
- semiparametric approach [Genest et al., 1995]

- **Bayesian methods:** [Smith, 2013]

- multivariate discrete data [Smith and Khaled, 2012]
- conditional copulae [Craiu and Sabeti, 2012]
- vine-copulae [Min and Czado, 2010]
- nonparametric approach [Wu et al., 2014]

The likelihood function is complicated!

In the continuous case, the density of a multivariate distribution, in its copula representation is

$$f(x|\lambda, \theta) = c(u; \theta) \prod_{j=1}^d f_j(x_j|\lambda_j)$$

where $u = (u_1, \dots, u_d) = (F_1(x_1; \lambda_1), \dots, F_d(x_d; \lambda_d))$.

The posterior distribution for (θ, λ) is

$$\pi(\theta, \lambda|x) \propto \pi(\theta, \lambda) \prod_{i=1}^n \left[c(u_i; \theta) \prod_{j=1}^d f(x_{ij}; \lambda_j) \right].$$

Remark: the likelihood function is not separable in $\lambda_1, \dots, \lambda_d$ and θ because u_i depends on the marginal parameter λ .

Why a semiparametric approach?

- if the interest is in a **functional of the dependence**, the likelihood function for it may be very complicated

Example: Clayton copula

- likelihood: $\mathfrak{L}(\theta; u, v) = \prod_{i=1}^n (\theta + 1)(u_i v_i)^{-(\theta+1)}(u_i^{-\theta} + v_i^{-\theta} - 1)^{-\frac{2\theta+1}{\theta}}$
- functionals: $\tau = \frac{\theta}{\theta+2}$ *and* $\lambda_L = 2^{-\frac{1}{\theta}}$ *but* $\rho = \dots$
- methods of selection of the copula may be unreliable

In this situation we derive an **approximated posterior distribution**

$$\pi(\phi|x) \propto \pi(\phi) L_{BEL}(\phi; x)$$

ABSCop Step 1: Marginal Estimation

Given a sample $X = (X_1, X_2, \dots, X_d)$ with joint cdf $F_X(x)$ and marginal cdf's $F_1(x_1; \lambda_1), \dots, F_d(x_d; \lambda_d)$

For $j = 1, \dots, d$

Derive a posterior sample for λ_j : $(\lambda_j^1, \dots, \lambda_j^{S_j})$ approximating the marginal posterior $\pi(\lambda_j | x_j)$

End For

ABSCop Step 2: Joint Estimation

For $b = 1, \dots, B$

- Draw $\phi^{(b)} \sim \pi(\phi)$
- Sample one value λ^{s_j} from each marginal posterior sample:
 $\lambda' = (\lambda_1^{(s_1)}, \dots, \lambda_d^{(s_d)})$
- Derive a matrix of uniformly distributed pseudo-data $u_{ij} = F_j(x_{ij}; \lambda_j^{(s_j)})$

$$u' = \begin{pmatrix} u_{11}^{(s_1)} & u_{12}^{(s_2)} & \dots & u_{1d}^{(s_d)} \\ u_{21}^{(s_1)} & u_{22}^{(s_2)} & \dots & u_{2d}^{(s_d)} \\ \dots & \dots & u_{ij}^{(s_j)} & \dots \\ u_{n1}^{(s_1)} & u_{n2}^{(s_2)} & \dots & u_{nd}^{(s_d)} \end{pmatrix}.$$

- Compute $L_{BEL}(\phi^{(b)}; u') = \omega_b$

End For

Validation: nonparametric estimation of the marginals

Suppose $(X_{11}, X_{21}, \dots, X_{d1}), \dots, (X_{1n}, X_{2n}, \dots, X_{dn})$ are independent random vectors with distribution function \mathbf{F} and marginal F_1, F_2, \dots, F_d .

The empirical estimator of the copula function

$C(u_1, u_2, \dots, u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_d^{-1}(u_d))$ is

$$C_n(u_1, u_2, \dots, u_d) = \mathbf{F}_n(F_{1n}^{-1}(u_1), F_{2n}^{-1}(u_2), \dots, F_{dn}^{-1}(u_d)),$$

where $\mathbf{F}_n, F_{1n}, F_{2n}, \dots, F_{dn}$ are the joint and marginal empirical distribution functions of the observations.

The empirical copula process is defined as

$$\mathbb{C}_n = \sqrt{n}(C_n - C)$$

and if the j -th first order partial derivative exists and is continuous on $V_{d,j} = \{u \in [0, 1]^d : 0 < u_j < 1\}$, then \mathbb{C}_n converges weakly to the Gaussian process $\{\mathbb{G}_C(u_1, u_2, \dots, u_d), 0 < u_1, u_2, \dots, u_d < 1\}$ in $\ell^\infty([0, 1]^d)$.

Goal: *estimating a functional of the dependence (Spearman's ρ , Kendall's τ , tail dependence coefficients λ_L and λ_U , etc.)*

- Select a quantity of interest ϕ and a prior $\pi(\phi)$

$$\rho = 12 \int_0^1 \int_0^1 C(u_j, u_h) du_j du_h - 3.$$

with $\pi(\rho) \sim \mathcal{U}(-1, 1)$.

- Select a (nonparametric) estimators ϕ_n

$$\rho_n = \frac{1}{n} \sum_{i=1}^n \left(\frac{12}{n^2 - 1} R_i Q_i \right) - 3 \frac{n+1}{n-1},$$

- Compute the empirical likelihood of ϕ based on its estimate
- Derive via simulation the posterior distribution $\pi(\phi; \mathbf{x})$

Clayton and Frank, $d = 2$

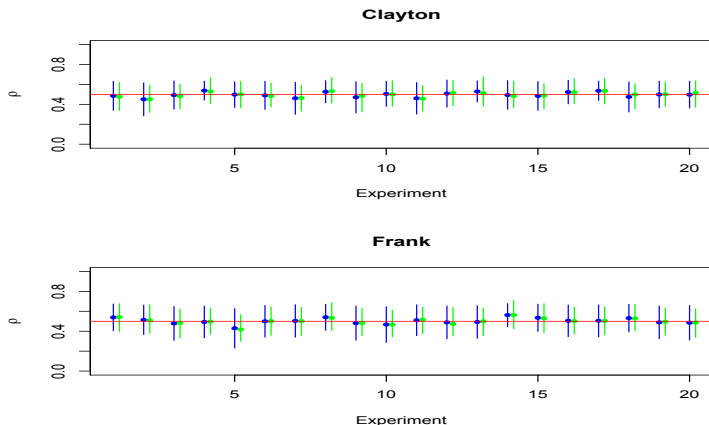


Figure: Comparison between frequentist (blue) and Bayesian estimates (green). 20 out of 500 experiments with simulations from a Clayton copula (above) and a Frank copula (below) ($n = 1000$).

What happens when $\rho \rightarrow 1$

Borkowf [2002] shows that the asymptotic variance of ρ_n is

$$\sigma^2(\rho_n) = 144(-9\theta_1^2 + \theta_2 + 2\theta_3 + 2\theta_4 + 2\theta_5), \quad (1)$$

where

$$\theta_1 = \mathbb{E}[F_1(X_1)F_2(Y_1)]$$

$$\theta_2 = \mathbb{E}[(1 - F_1(X_1))^2(1 - F_2(Y_1))^2]$$

$$\theta_3 = \mathbb{E}[(1 - F(X_1, Y_2))(1 - F(X_2))(1 - F(Y_1))]$$

$$\theta_4 = \mathbb{E}[(1 - F_1(\max\{X_1, X_2\}))(1 - F_2(Y_1))(1 - F_2(Y_2))]$$

$$\theta_5 = \mathbb{E}[(1 - F_1(X_1))(1 - F_1(X_2))(1 - F_2(\max\{Y_1, Y_2\}))].$$

Consistent estimates of the above quantities are available in Genest & Favre [2007].

However, in the case of **perfect rank agreement**, when plugging-in the sample estimates of the θ_j 's into expression (1), one gets a **negative number**.

Intervals length for $d = 2$

Table: Simulations from different copulas: average length and empirical coverage based of the intervals obtained both via frequentist and Bayesian methods, based on 500 repetitions of the experiment

		Ave. Length	Coverage
Clayton ($\rho = 0.5$)	<i>Freq.</i>	0.2664	0.998
	<i>Bayes.</i>	0.2597	1.000
Frank ($\rho = 0.5$)	<i>Freq.</i>	0.3172	1.000
	<i>Bayes.</i>	0.2735	1.000
Gumbel ($\rho = 0.68$)	<i>Freq.</i>	-	-
	<i>Bayes.</i>	0.2966	1.000
Gaussian ($\rho = 0.8$)	<i>Freq.</i>	-	-
	<i>Bayes.</i>	0.2931	1.000

Comparison with parametric methods

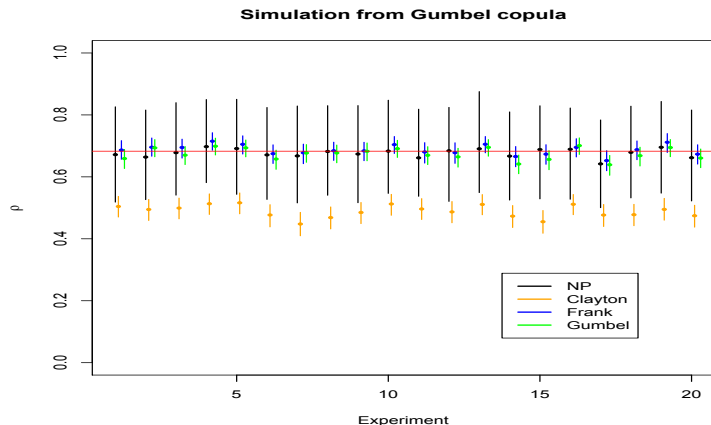


Figure: Bayesian point estimates (points) and credible intervals for 20 out of 500 experiments with data from a Gumbel copula with $\theta = 2$, obtained by specifying a Clayton model (orange), a Frank model (blue) and a Gumbel model (green) or by using our semiparametric approach (black).

The **upper and lower tail dependence indices** are defined

$$\lambda_U = \lim_{u \rightarrow 1} \Pr\{X_i > F_i^{-1}(u) | X_j > F_j^{-1}(u)\}$$

$$\lambda_L = \lim_{v \rightarrow 1} \Pr\{X_i \leq F_i^{-1}(v) | X_j \leq F_j^{-1}(v)\}$$

but may be rewritten in terms of copulas

$$\lambda_U = \lim_{v \rightarrow 1} \frac{1 - 2v - C(v, v)}{1 - v}, \quad \lambda_L = \lim_{v \rightarrow 0} \frac{C(v, v)}{v}.$$

that may be estimated by [Joe et al., 1992]

$$\hat{\lambda}_U = 2 - \frac{n}{k} \left\{ 1 - \hat{C}_n \left(\frac{n-k}{n}, \frac{n-k}{n} \right) \right\}, \quad \hat{\lambda}_L = \frac{n}{k} \hat{C}_n \left(\frac{k}{n}, \frac{k}{n} \right)$$

Schmidt and Stadtmüller [2006] prove

- strong consistency
- asymptotic normality for these estimators.
- derive the asymptotic variance

Clayton, $d=2$

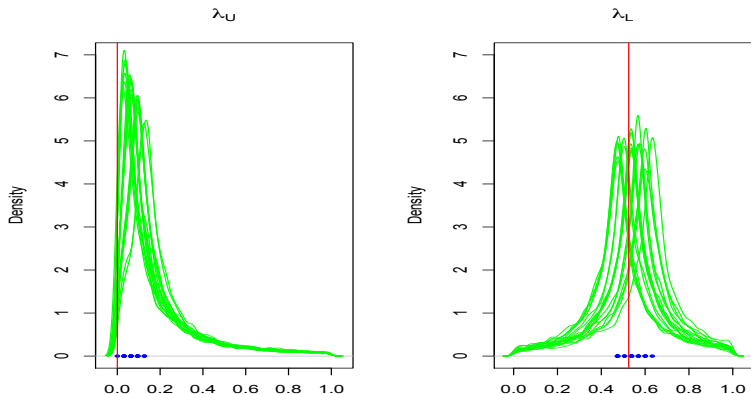


Figure: Comparison between frequentist (blue) and Bayesian (green) estimates for λ_U (left) and λ_L (right). 20 out of 500 simulations from a Clayton copula with $\theta = 1.076$ ($n = 1000$). The true values are $\lambda_U^{true} = 0$ and $\lambda_L^{true} = 2^{-\frac{1}{\theta}}$ (red lines).

Goal: *estimating a functional of the dependence (Spearman's ρ , Kendall's τ , tail dependence coefficients λ_L and λ_U , etc.)*

- Select a quantity of interest ϕ and a prior $\pi(\phi)$

$$\rho_1 = \frac{\int_{[0,1]^d} (C(u) - \Pi(u)) du}{\int_{[0,1]^d} (M(u) - \Pi(u)) du} = h(d) \left\{ 2^d \int_{[0,1]^d} C(u) du - 1 \right\},$$

where $h(d) = (d+1)/\{2^d - (d+1)\}$ or

$$\rho_2 = h(d) \left\{ 2^d \int_{[0,1]^d} \Pi(u) dC(u) - 1 \right\}.$$

- Select a (nonparametric) estimators ϕ_n

$$\hat{\rho}_{1n} = h(d) \left\{ 2^d \int_{[0,1]^d} \hat{C}_n(u) du - 1 \right\} = h(d) \left\{ \frac{2^d}{n} \sum_{i=1}^n \prod_{j=1}^d (1 - \hat{U}_{ij}) - 1 \right\}$$

$$\hat{\rho}_{2n} = h(d) \left\{ 2^d \int_{[0,1]^d} \Pi(u) d\hat{C}_n(u) - 1 \right\} = h(d) \left\{ \frac{2^d}{n} \sum_{i=1}^n \prod_{j=1}^d \hat{U}_{ij} - 1 \right\}.$$

Asymptotic properties of these estimators are explored and assessed in Schmid and Schmidt [2007]. In particular it is known that

$$\sqrt{n}(\hat{\rho}_{kn} - \rho_k) \overset{\sim}{\sim} \mathcal{N}(0, \sigma_k^2), \quad k = 1, 2.$$

Clayton

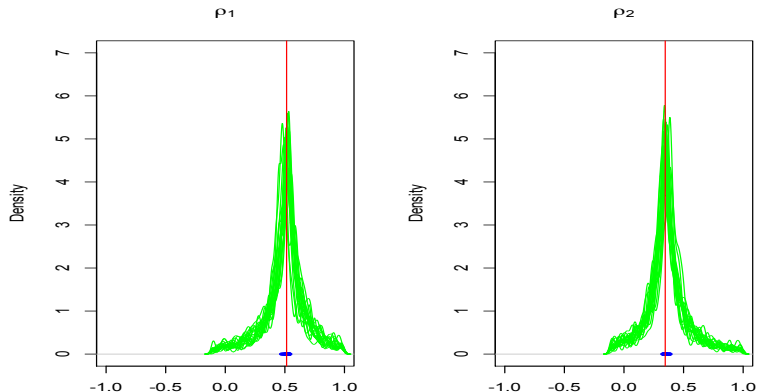


Figure: Comparison between frequentist (blue) and Bayesian (green) estimates of ρ_1 (left) and ρ_2 (right). 20 out of 500 experiments with simulation from a Clayton copula with $\theta = 1.076$ ($n = 1000$).

Table: Average lengths of the confidence intervals (based on a bootstrap estimator of the variance of the estimates) and of the corresponding Bayesian credible intervals obtained in 50 repetitions of each experiment of dimension d by simulating data from a Clayton copula with $\theta = 1.076$.

	$\hat{\rho}_1^{freq}$	$\hat{\rho}_2^{freq}$	$\hat{\rho}_1^{Bayes}$	$\hat{\rho}_2^{Bayes}$
$d = 2$	0.0032	0.0032	1.1933	1.1801
$d = 3$	0.0026	0.0026	1.0844	1.0853
$d = 4$	0.0026	0.0026	0.9495	0.9594
$d = 5$	0.0027	0.0027	0.8728	0.8914
$d = 6$	0.0027	0.0027	0.8211	0.8224
$d = 7$	0.0030	0.0030	0.8022	0.7882
$d = 8$	0.0031	0.0031	0.7828	0.7541
$d = 9$	0.0032	0.0032	0.7680	0.7492
$d = 10$	0.0035	0.0035	0.7558	0.7439
$d = 25$	0.0047	0.0047	0.7462	0.7480
$d = 50$	0.0073	0.0073	0.7299	0.7634

GARCH(1,1) for Student- t innovation I

Real dataset containing the log-returns FTSE-MIB of five Italian financial institutes

- Monte dei Paschi di Siena
- Banco Popolare
- Unicredit
- Intesa-Sanpaolo
- Mediobanca

by assuming that the log-returns for each bank may be modelled as a **generalized autoregressive conditional heteroscedastic model with parameters** (1,1) and Student- t innovations.

Data refers to weekdays from 01/07/2013 to 30/06/2014 available on the web-page <https://it.finance.yahoo.com>.

GARCH(1,1) for Student- t innovation II

For $t = 1, \dots, T$,

$$\begin{aligned}y_t &= \varepsilon_t \sqrt{\frac{\nu-2}{\nu}} \omega_t h_t; \\h_t &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta h_{t-1}; \\ \varepsilon_t &\sim \mathcal{N}(0, 1); \\ \omega_t &\sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right),\end{aligned}$$

where $\alpha_0 > 0$, $\alpha_1, \beta \geq 0$, $\nu > 2$ and $IG(a, b)$ denotes the invert gamma distribution with shape parameter a and scale parameter b .

Log-returns

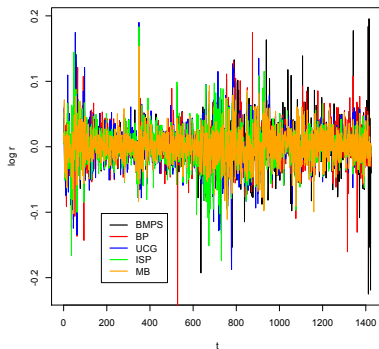


Figure: Log-returns of Monte dei Paschi di Siena (BMPS), Banco Popolare (BP), Unicredit (UCG), Intesa-Sanpaolo (ISP) and Mediobanca (MB) from 01/07/2013 to 30/06/2014, available on the web-page <https://it.finance.yahoo.com>

Posterior Distribution of ρ

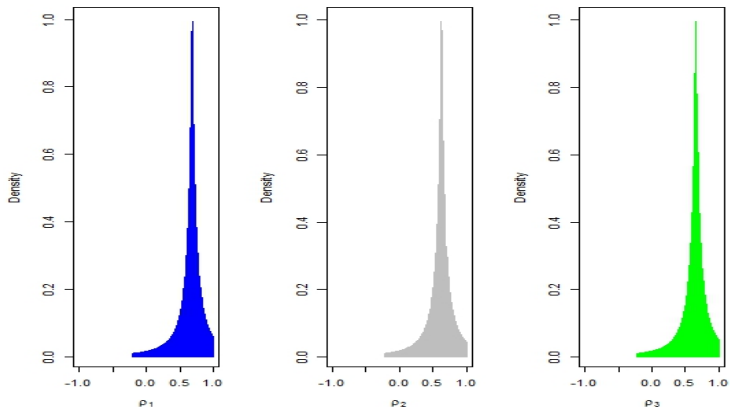


Figure: Approximation of the posterior distribution of the Spearman's ρ for the log-returns of the investments of five Italian institutes based on 10,000 simulations.

The g -and- k distribution is a popular example of a quantile distribution. This is a transformation of the standard normal distribution function, as follows:

$$Q(z(p); \theta) = a + b \left(1 + c \frac{1 - \exp(-gz(p))}{1 + \exp(-gz(p))} \right) (1 + z(p)^2)^k z(p)$$

where $\theta = (a, b, g, k)$ and c is commonly set fixed at 0.8.

Quantile distributions II

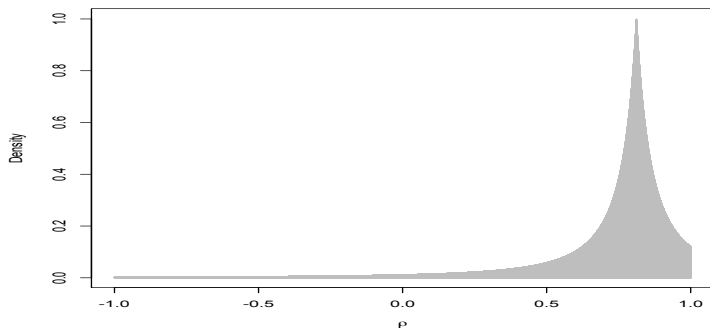


Figure: Spearman's ρ approximated posterior distribution by assuming marginal quantile distributions.

Posterior Distribution of λ_L and λ_U

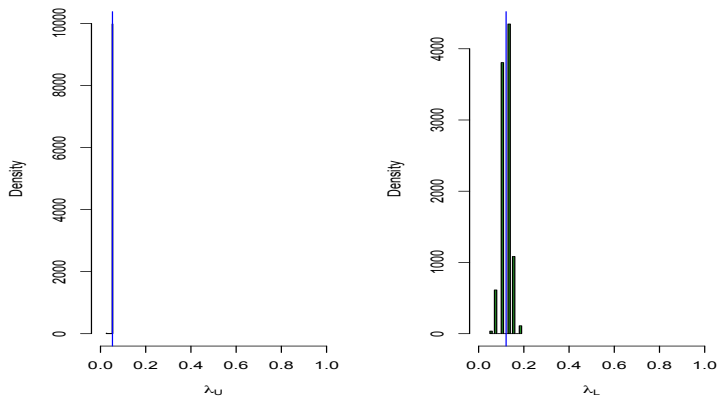


Figure: Approximation of the posterior distribution of the Spearman's ρ for the log-returns of the investments of five Italian institutes based on 10,000 simulations.

Conditional copulas

A biased estimation of the conditional ρ is

$$\hat{\rho}_n(x) = 12 \sum_{i=1}^n w_{ni}(x, h_n)(1 - \hat{U}_{1i})(1 - \hat{U}_{2i}) - 3$$

- bootstrap likelihood
[Zhu et al., 2015]
- bootstrap (unbiased) estimator
[Lemyre & Quesy, 2016]

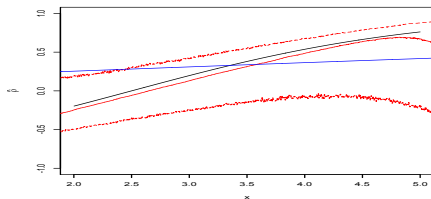


Figure: Simulations from the conditional Clayton copula, true function ρ in black, Bayesian estimates in red, frequentist in blue.

The method presents some advantages and disadvantages.

- ease of elicitation
- robustness in terms of model miss-specification
- generality
- in practical applications, there are often available only asymptotically unbiased estimators
- inefficient with respect to parametric methods (under the assumption that the chosen model is the true one)

Thank you for your attention!

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