Assessing the dependence of high-dimensional time series via sample autocovariances and correlations

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Joint work with
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Motivation: S&P 500 Index

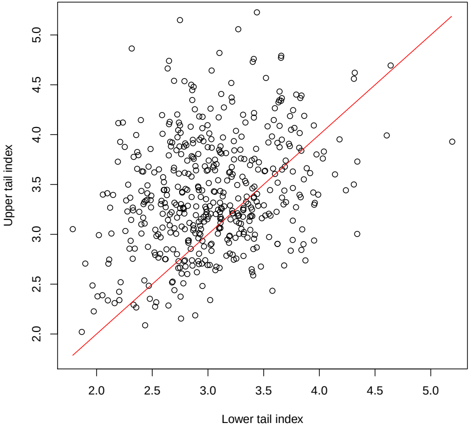

Figure: Estimated tail indices of log-returns of 478 time series in the S&P 500 index.

Setup

- **Data matrix** $\boldsymbol{X} = \boldsymbol{X}_p$: $p \times n$ matrix with iid centered columns.

$$\boldsymbol{X} = (X_{it})_{i=1,\ldots,p;t=1,\ldots,n}$$

- **Sample covariance matrix** $\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'$
- **Ordered eigenvalues** of $\boldsymbol{S}$

$$\lambda_1(\boldsymbol{S}) \geq \lambda_2(\boldsymbol{S}) \geq \cdots \geq \lambda_p(\boldsymbol{S})$$

- **Applications:**
  - Principal Component Analysis
  - Linear Regression, . . .

Sample Correlation Matrix

- **Sample correlation matrix $\mathbf{R}$** with entries

$$R_{ij} = \frac{S_{ij}}{\sqrt{S_{ii}S_{jj}}}, \quad i,j = 1,\ldots,p$$

and eigenvalues

$$\lambda_1(\mathbf{R}) \ge \lambda_2(\mathbf{R}) \ge \cdots \ge \lambda_p(\mathbf{R}).$$

Data structure:

$$\boldsymbol{X}_p = \mathbf{A}_p \mathbf{Z}_p \,,$$

where $\mathbf{A}_p$ is a deterministic $p \times p$ matrix such that $(\|\mathbf{A}_p\|)$ is bounded and

$$\mathbf{Z}_p = (Z_{it})_{i=1,\ldots,p;t=1,\ldots,n}$$

has iid, centered entries with unit variance (if finite).

The Model

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- **Population covariance matrix $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}'$.**
- **Population correlation matrix**

$$\boldsymbol{\Gamma} = (\mathrm{diag}(\boldsymbol{\Sigma}))^{-1/2}\boldsymbol{\Sigma}(\mathrm{diag}(\boldsymbol{\Sigma}))^{-1/2}$$

- Note: $\mathbb{E}[\boldsymbol{S}] = \boldsymbol{\Sigma}$ but $\mathbb{E}[R_{ij}] = \Gamma_{ij} + O(n^{-1})$.

The Model

	Sample	**Population**
Covariance matrix	S	$\mathbf{\Sigma}$
Correlation matrix	$\mathbf{R}$	$\mathbf{\Gamma}$

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Covariance matrix	$\boldsymbol{S}$	$\boldsymbol{\Sigma}$
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Growth regime:

$$n = n_p \to \infty \quad \text{and} \quad \frac{p}{n_p} \to \gamma \in [0, \infty)\,, \quad \text{as } p \to \infty\,.$$

- High dimension: $\lim_{p\to\infty} \frac{p}{n} \in (0, \infty)$
- Moderate dimension: $\lim_{p\to\infty} \frac{p}{n} = 0$

Main Result

Approximation Under Finite Fourth Moment

Assume $\boldsymbol{X} = \mathbf{A}\mathbf{Z}$ and $\mathbb{E}[Z_{11}^4] < \infty$. Then we have as $p \to \infty$,

$$\sqrt{n/p}\, \| \operatorname{diag}(\boldsymbol{S}) - \operatorname{diag}(\boldsymbol{\Sigma}) \| \stackrel{a.s.}{\to} 0\,.$$

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Assume $\boldsymbol{X} = \mathbf{Z}$ and $\mathbb{E}[Z_{11}^4] = \infty$. Then we have as $p \to \infty$,

$$\underbrace{c_{np}}_{\to 0} \| \boldsymbol{S} - \operatorname{diag}(\boldsymbol{S}) \| \stackrel{a.s.}{\to} 0\,.$$

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Main result

Assume $\boldsymbol{X} = \mathbf{AZ}$ and $\mathbb{E}[Z_{11}^4] < \infty$. Then we have as $p \to \infty$,

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Relevance: Note that

$$\mathbf{R} = (\mathrm{diag}(\boldsymbol{S}))^{-1/2}\boldsymbol{S}\,(\mathrm{diag}(\boldsymbol{S}))^{-1/2}\,.$$

$\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'$ and $\mathbf{R} = \boldsymbol{Y}\boldsymbol{Y}'$, where

$$\boldsymbol{Y} = (Y_{ij})_{p\times n} = \left(\frac{X_{ij}}{\sqrt{\sum_{t=1}^n X_{it}^2}}\right)_{p\times n}$$

In general, any two entries of $\mathbf{Y}$ are dependent.

A Comparison Under Finite Fourth Moment

Approximation of the sample correlation matrix

Assume $\boldsymbol{X} = \mathbf{AZ}$ and $\mathbb{E}[Z_{11}^4] < \infty$. Then we have

$$\sqrt{\frac{n}{p}}\,\|\mathbf{R} - \underbrace{(\mathrm{diag}(\boldsymbol{\Sigma}))^{-1/2}\boldsymbol{S}(\mathrm{diag}(\boldsymbol{\Sigma}))^{-1/2}}_{\boldsymbol{S}^{\mathbf{Q}}}\| \stackrel{a.s.}{\to} 0\,.$$

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Spectrum comparison

An application of Weyl's inequality yields

$$\sqrt{\frac{n}{p}}\max_{i=1,\ldots,p}\left|\lambda_i(\mathbf{R}) - \lambda_i(\boldsymbol{S}^{\mathbf{Q}})\right| \le \sqrt{\frac{n}{p}}\,\|\mathbf{R} - \boldsymbol{S}^{\mathbf{Q}}\| \stackrel{a.s.}{\to} 0\,.$$

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Operator norm consistent estimation

$$\|\mathbf{R} - \boldsymbol{\Gamma}\| = O(\sqrt{p/n}) \quad \text{a.s.}$$

Notation

- **Empirical spectral distribution** of $p \times p$ matrix $\mathbf{C}$ with real eigenvalues $\lambda_1(\mathbf{C}), \ldots, \lambda_p(\mathbf{C})$:

$$F_{\mathbf{C}}(x) = \frac{1}{p} \sum_{i=1}^{p} \mathbb{1}_{\{\lambda_i(\mathbf{C}) \le x\}}, \qquad x \in \mathbb{R}.$$

- **Stieltjes transform**:

$$s_{\mathbf{C}}(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, \mathrm{d}F_{\mathbf{C}}(x) = \frac{1}{p} \operatorname{tr}(\mathbf{C} - z\mathbf{I})^{-1}, \quad z \in \mathbb{C}^+,$$

- **Limiting spectral distribution**:
  Weak convergence of $(F_{\mathbf{C}_p})$ to distribution function F a.s.

Limiting Spectral Distribution of $\mathbf{R}$

Assume $\boldsymbol{X} = \mathbf{AZ}$, $\mathbb{E}[Z_{11}^4] < \infty$ and that $F_{\mathbf{\Gamma}}$ converges to a probability distribution H.

1. If $p/n \to \gamma \in (0,\infty)$, then $F_{\mathbf{R}}$ converges weakly to a distribution function $F_{\gamma,H}$, whose Stieltjes transform s satisfies

$$s(z) = \int \frac{\mathrm{d}H(t)}{t(1-\gamma-\gamma s(z)) - z}\,, \quad z \in \mathbb{C}^+\,.$$

2. If $p/n \to 0$, then $F_{\sqrt{n/p}(\mathbf{R}-\mathbf{\Gamma})}$ converges weakly to a distribution function F, whose Stieltjes transform s satisfies

$$s(z) = -\int \frac{\mathrm{d}H(t)}{z + t\widetilde{s}(z)}\,, \quad z \in \mathbb{C}^+\,,$$

where $\widetilde{s}$ is the unique solution to $\widetilde{s}(z) = -\int (z + t\widetilde{s}(z))^{-1} t\,\mathrm{d}H(t)$ and $z \in \mathbb{C}^+$.

Special Case $\mathbf{A} = \mathbf{I}$

Simplified assumptions:

1. iid, symmetric entries $X_{it} \stackrel{\mathrm{d}}{=} X$
2. **Growth regime:** $\lim_{p\to\infty} \frac{p}{n} = \gamma \in [0, 1]$

Marčenko–Pastur and Semicircle Law

- **Marčenko–Pastur law** F_γ has density

$$f_\gamma(x) = \begin{cases} \frac{1}{2\pi x\gamma}\sqrt{(b-x)(x-a)}\,, & \text{if } x \in [a,b], \\ 0\,, & \text{otherwise,} \end{cases}$$

where $a = (1-\sqrt{\gamma})^2$ and $b = (1+\sqrt{\gamma})^2$.

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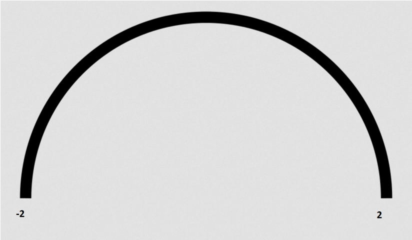
- **Semicircle law** SC



Extreme Eigenvalues

Largest and smallest eigenvalues of $\mathbf{R}$

If $p/n \to \gamma \in [0,1]$ and $\mathbb{E}[X^4] < \infty$, then

$$\sqrt{n/p}\,(\lambda_1(\mathbf{R}) - 1) \stackrel{a.s.}{\to} 2 + \sqrt{\gamma}$$

and

$$\sqrt{n/p}\,(\lambda_p(\mathbf{R}) - 1) \stackrel{a.s.}{\to} -2 + \sqrt{\gamma}\,.$$

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- Earlier: $\|\mathbf{R} - \mathbf{\Gamma}\| = O(\sqrt{p/n})\,\mathrm{a.s.}$
- In this case:

$$\sqrt{n/p}\,\|\mathbf{R} - \mathbf{\Gamma}\| \stackrel{a.s.}{\to} 2 + \sqrt{\gamma}\,.$$

Limiting Spectral Distribution

Marčenko–Pastur Theorem

Assume $\mathbb{E}[X^2] = 1$. Then $(F_{\boldsymbol{S}})$ converges weakly to F_γ.
If $\mathbb{E}[X^4] < \infty$ and $p/n \to 0$, then $(F_{\sqrt{n/p}(\boldsymbol{S}-\mathbf{I})})$ converges weakly to SC.

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If $\mathbb{E}[X^4] < \infty$ and $p/n \to 0$, then $(F_{\sqrt{n/p}(\boldsymbol{S}-\mathbf{I})})$ converges weakly to SC.

JH (2018+)

Under the domain of attraction type-condition for the Gaussian law,

$$\lim_{p\to\infty} \frac{n}{p} n\mathbb{E}\left[Y_{11}^4\right] = 0\,,$$

the sequence $(F_{\mathbf{R}})$ converges weakly to F_γ.
If in addition $p/n \to 0$, then $(F_{\sqrt{n/p}(\mathbf{R}-\mathbf{I})})$ converges weakly to SC.

Here $Y_{ij} = \frac{X_{ij}}{\sqrt{\sum_{t=1}^n X_{it}^2}}$.

Simulation Study

- **Regular variation** with index $\alpha > 0$:

$$\mathbb{P}(|X| > x) = x^{-\alpha} L(x),$$

  where L is a slowly varying function.
- This implies $\mathbb{E}[|X|^{\alpha+\varepsilon}] = \infty$ for any $\varepsilon > 0$.

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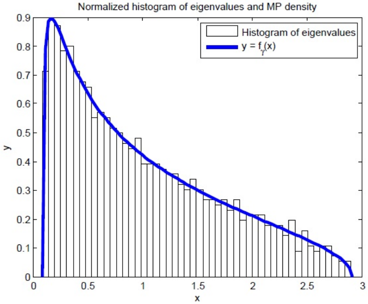
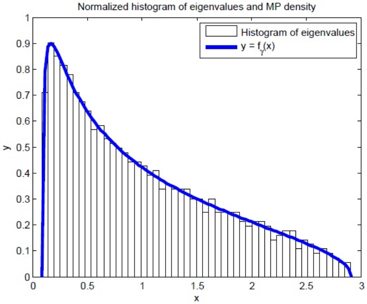
- Procedure:
  1. Simulate $\boldsymbol{X}$
  2. Plot histograms of $(\lambda_i(\mathbf{R}))$ and $(\lambda_i(\boldsymbol{S}))$
  3. Compare with Marčenko–Pastur density

$\alpha = 6$



(a) Sample correlation



(b) Sample covariance

$$\alpha = 6, n = 2000, p = 1000$$

Infinite Fourth Moment

- **Regular variation** with index $\alpha \in (0,4)$
- **Normalizing sequence** (a_{np}^2) such that

$$np\,\mathbb{P}(X^2 > a_{np}^2 x) \to x^{-\alpha/2}, \quad \text{as } n \to \infty \text{ for } x > 0.$$

Then $a_{np} = (np)^{1/\alpha}\ell(np)$ for a slowly varying function ℓ.

Reduction to Diagonal

Diagonal

$\boldsymbol{X}$ with iid regularly varying entries $\alpha \in (0,4)$ and $p = n^{\beta}\ell(n)$ with $\beta \in [0,1]$. We have

$$a_{np}^{-2} \|\boldsymbol{X}\boldsymbol{X}' - \mathrm{diag}(\boldsymbol{X}\boldsymbol{X}')\| \xrightarrow{\mathbb{P}} 0\,,$$

where $\|\cdot\|$ denotes the spectral norm.

$$(\boldsymbol{X}\boldsymbol{X}')_{ij} = \sum_{t=1}^{n} X_{it}X_{jt}.$$

- **Weyl's inequality**

$$\max_{i=1,\ldots,p} \left|\lambda_i(\mathbf{A}+\mathbf{B}) - \lambda_i(\mathbf{A})\right| \le \|\mathbf{B}\|\,.$$

- Choose $\mathbf{A}+\mathbf{B} = \boldsymbol{X}\boldsymbol{X}'$ and $\mathbf{A} = \mathrm{diag}(\boldsymbol{X}\boldsymbol{X}')$ to obtain

$$a_{np}^{-2} \max_{i=1,\ldots,p} \left|\lambda_i(\boldsymbol{X}\boldsymbol{X}') - \lambda_i(\mathrm{diag}(\boldsymbol{X}\boldsymbol{X}'))\right| \stackrel{\mathbb{P}}{\to} 0\,, \quad n \to \infty\,.$$

- **Note:** Limit theory for $(\lambda_i(\boldsymbol{S}))$ reduced to (S_{ii}).

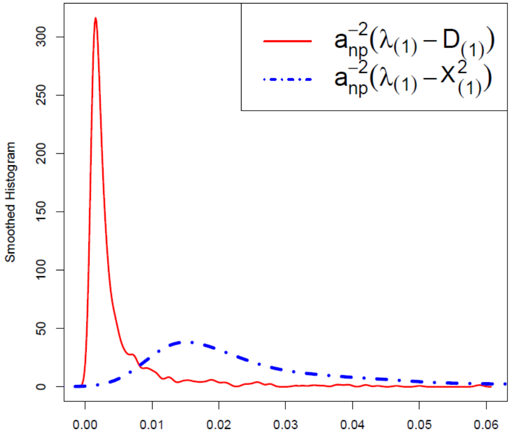


Figure: Smoothed histogram based on 20000 simulations of the approximation error for the normalized eigenvalue $a_{np}^{-2}\lambda_1(\boldsymbol{S})$ for entries X_{it} with $\alpha = 1.6$, $\beta = 1$, $n = 1000$ and $p = 200$.

- $\mathbf{v}_k$ unit eigenvector of $\boldsymbol{S}$ associated to $\lambda_k(\boldsymbol{S})$
- Unit eigenvectors of $\mathrm{diag}(\boldsymbol{S})$ are canonical basisvectors $\mathbf{e}_j$.

Eigenvectors

$\boldsymbol{X}$ with iid regularly varying entries with index $\alpha \in (0,4)$ and $p_n = n^\beta \ell(n)$ with $\beta \in [0,1]$. Then for any fixed $k \geq 1$,

$$\|\mathbf{v}_k - \mathbf{e}_{L_k}\|_{\ell_2} \stackrel{\mathbb{P}}{\to} 0\,, \quad n \to \infty\,.$$

Localization vs. Delocalization

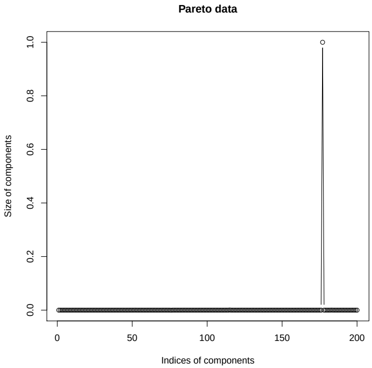


Figure: $X \sim$ Pareto(0.8)

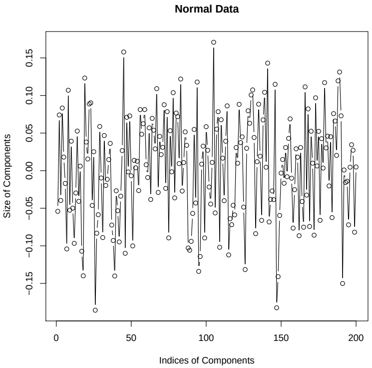


Figure: $X \sim N(0, 1)$

Components of eigenvector $\mathbf{v}_1$. $p = 200$, $n = 1000$.

Point Process of Normalized Eigenvalues

Point process convergence

$$N_n = \sum_{i=1}^{p} \delta_{a_{np}^{-2}\lambda_i(\boldsymbol{X}\boldsymbol{X}')} \xrightarrow{\text{d}} \sum_{i=1}^{\infty} \delta_{\Gamma_i^{-2/\alpha}} = N$$

The limit is a PRM on $(0, \infty)$ with mean measure $\mu(x, \infty) = x^{-\alpha/2}, x > 0$, and

$$\Gamma_i = E_1 + \cdots + E_i\,, \quad (E_i) \text{ iid standard exponential.}$$

Point Process of Normalized Eigenvalues

- **Limiting distribution:** For $k \geq 1$,

$$\lim_{n\to\infty} \mathbb{P}(a_{np}^{-2}\lambda_k \leq x) = \lim_{n\to\infty} \mathbb{P}(N_n(x,\infty) < k) = \mathbb{P}(N(x,\infty) < k)$$
$$= \sum_{s=0}^{k-1} \frac{\left(x^{-\alpha/2}\right)^s}{s!} \mathrm{e}^{-x^{-\alpha/2}}, \quad x > 0\,.$$

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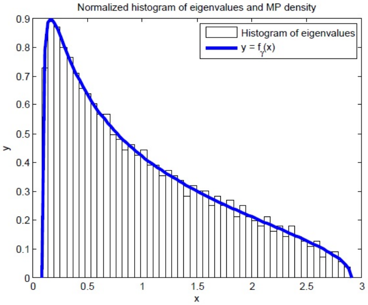
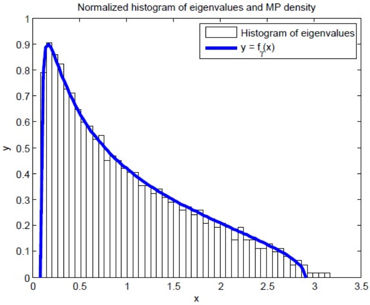
- **Largest eigenvalue**

$$\frac{n}{a_{np}^2}\lambda_1(\boldsymbol{S}) \xrightarrow{\mathrm{d}} \Gamma_1^{-\alpha/2}\,,$$

where the limit has a *Fréchet distribution* with parameter $\alpha/2$.
Soshnikov (2006), Auffinger et al. (2009), Auffinger and Tang (2016), Davis et al. (2014, 2016[2]), JH and Mikosch (2016)



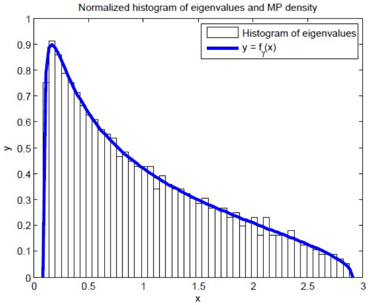
(a) Sample correlation



(b) Sample covariance

$$\alpha = 3.99, n = 2000, p = 1000$$

$\alpha = 3$



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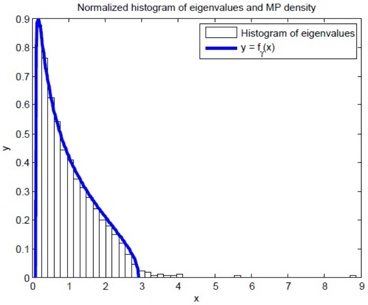
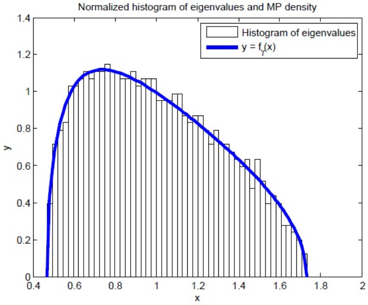


(b) Sample covariance

$$\alpha = 3, n = 2000, p = 1000$$

$\alpha = 2.1$



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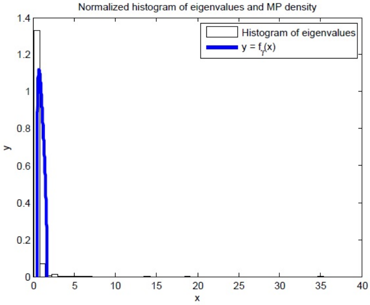

(b) Sample covariance

$$\alpha = 2.1, n = 10000, p = 1000$$

(Z_{it}): iid field of regularly varying random variables.

- **Stochastic volatility model**:

$$\boldsymbol{X} = \big(Z_{it}\,\sigma_{it}^{(n)}\big)_{p\times n}$$

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- **Generate deterministic covariance structure $\mathbf{A}$**:

$$\boldsymbol{X} = \mathbf{A}^{1/2}\mathbf{Z}$$

Davis et al. (2014)

(Z_{it}): iid field of regularly varying random variables.

- **Dependence among rows and columns**:

$$X_{it} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} h_{kl} Z_{i-k,t-l}$$

with some constants h_{kl}. Davis et al. (2016)

Heavy Tails and Dependence

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- **Relation to iid case:**

$$\boldsymbol{X}\boldsymbol{X}' = \sum_{l_1,l_2=0}^{\infty} \sum_{k_1,k_2=0}^{\infty} h_{k_1 l_1} h_{k_2 l_2} \mathbf{Z}(k_1,l_1)\mathbf{Z}'(k_2,l_2)\,,$$

  where

$$\mathbf{Z}(k,l) = (Z_{i-k,t-l})_{i=1,\ldots,p;t=1,\ldots,n}\,, \quad l,k \in \mathbb{Z}\,.$$

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where

$$\mathbf{Z}(k,l) = (Z_{i-k,t-l})_{i=1,\ldots,p;t=1,\ldots,n}\,, \quad l,k \in \mathbb{Z}\,.$$

- **Location of squares:**

$$\boldsymbol{M}_{ij} = \sum_{l \in \mathbb{Z}} h_{il} h_{jl}, \qquad i,j \in \mathbb{Z}\,.$$

Autocovariance Matrices

- For $s \geq 0$,

$$\boldsymbol{X}_n(s) = (X_{i,t+s})_{i=1,\ldots,p;\, t=1,\ldots,n}\,, \quad n \geq 1\,.$$

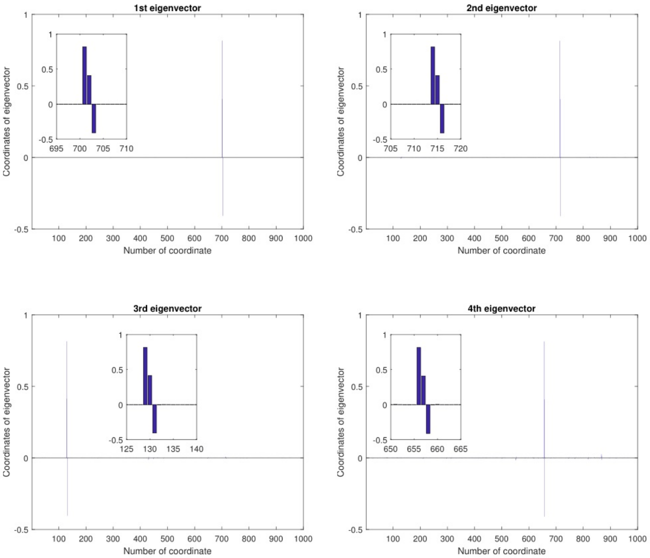
Then $\boldsymbol{X}_n = \boldsymbol{X}_n(0)$.

- **Autocovariance matrix** for lag s

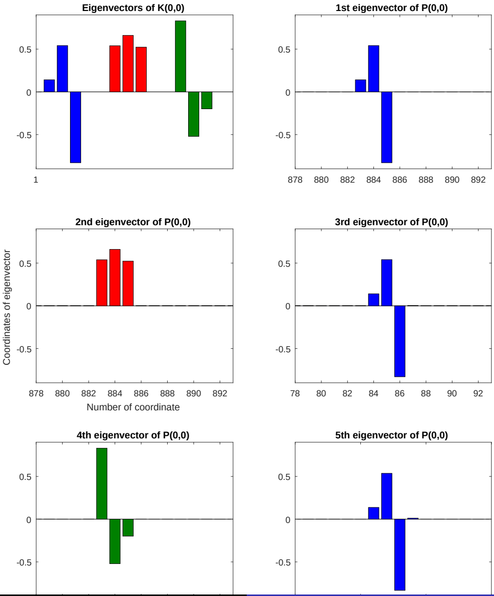
$$\boldsymbol{X}_n(0)\boldsymbol{X}_n(s)'$$

- Limit theory for singular values of such matrices.

Autocovariance eigenvectors



Autocovariance eigenvectors



Thank you!