Impulse Control: Recent Progress and Applications

Christoph Belak
Technische Universität Berlin

Joint work with **Sören Christensen** (Kiel), **Lukas Mich** (Trier), and **Frank Seifried** (Trier).

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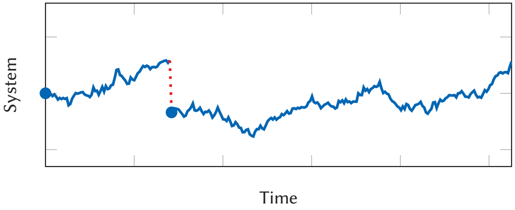

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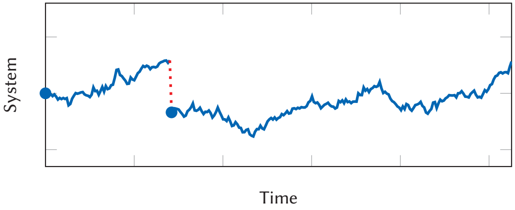

Applications: Harvesting, inventory control, real options, control of exchange rates, optimal investment with transaction costs, ...

Outline

(1) Impulse Control: General Formulation

(2) Superharmonic Functions and Stochastic Perron

(3) Optimal Investment with Transaction Costs

(4) Numerical Results

Impulse Control: General Formulation

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$ as follows:

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\begin{aligned}
\mathrm{d}X(t) &= \mu\big(X(t)\big)\mathrm{d}t + \sigma\big(X(t)\big)\,\mathrm{d}W(t), \qquad t \in [\tau_k, \tau_{k+1}),\\
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where

- the stopping times τ_k are **increasing** and **do not accumulate** in that

$$\mathbb{P}\big[\lim_{k\to\infty} \tau_k > T\big] = 1,$$

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The **objective** is to find a maximizer of

$$\mathcal{V}(t,x) = \sup_{\Lambda\in\mathcal{A}(t,x)} \mathbb{E}\Big[\sum_{k\in\mathbb{N}} K\big(X^\Lambda_{t,x}(\tau_k-), \Delta_k\big)\mathbb{1}_{\{\tau_k\le T\}} + g\big(X^\Lambda_{t,x}(T)\big)\Big].$$

The Quasi-Variational Inequalities

Classical Theory: Compute the value function $\mathcal{V}$ by solving

$$\min\{-\partial_t\mathcal{V}(t,x) - \mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\} = 0,$$
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and $\mathcal{M}$ is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta \in Z(x)} \left[\mathcal{V}\big(t, \Gamma(x,\Delta)\big) + K(x,\Delta)\right].$$

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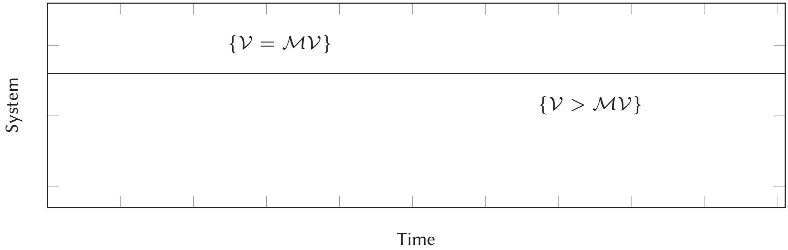

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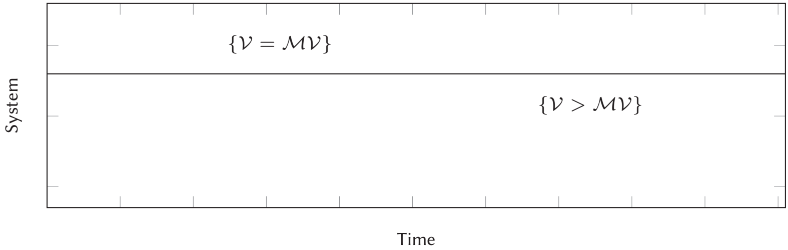

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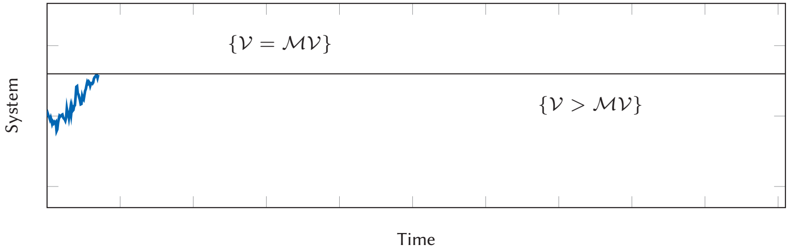

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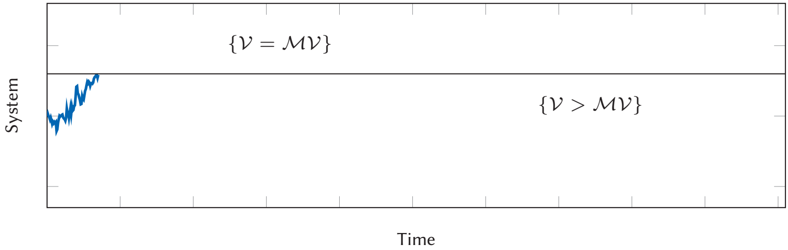

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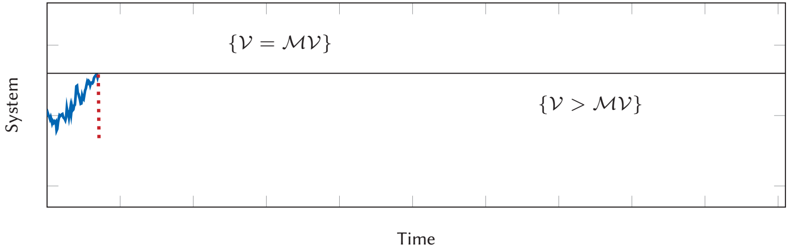

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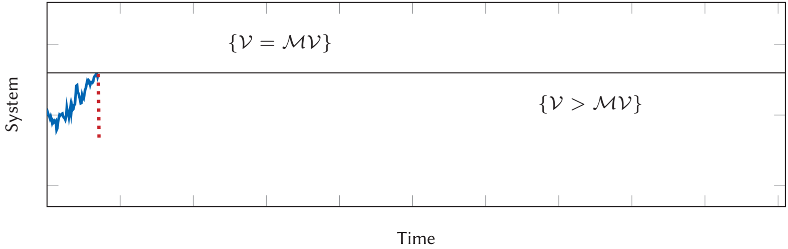

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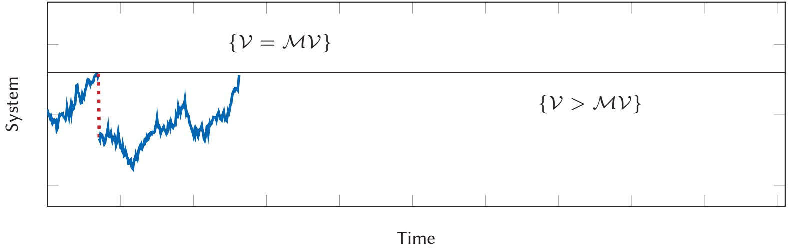

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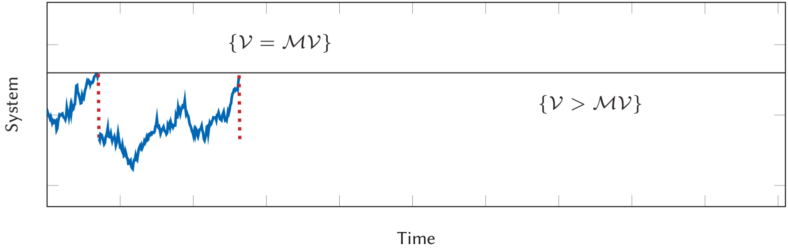

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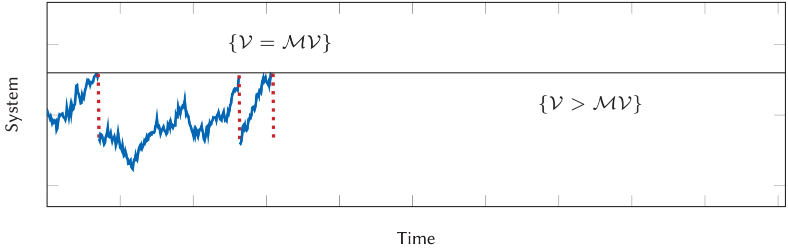

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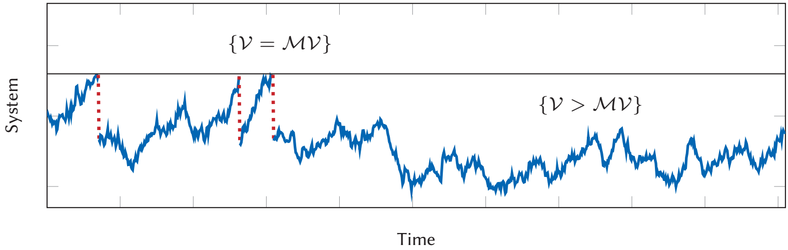

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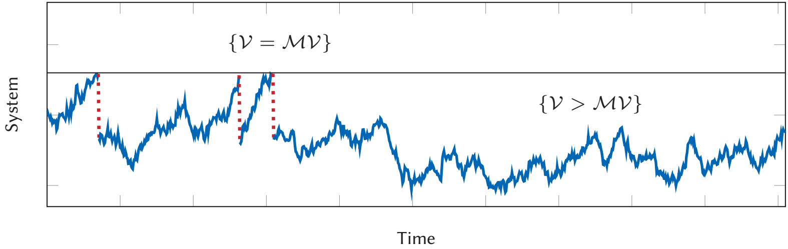

Problem: Verification requires a solution of the QVIs which is sufficiently smooth to apply Itō's formula.

Superharmonic Functions and Stochastic Perron

An Implicit Optimal Stopping Problem

By the (heuristic!) **Bellman principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, \bar{X}(\tau)\big)\big]$$

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Typically: $\mathcal{M}\mathcal{V}$ is upper semicontinuous if $\mathcal{V}$ is upper semicontinuous. So we essentially need $\mathcal{V}$ to be continuous.

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Proof: Iteratively solve the implicit optimal stopping problem. The argument adapts classical optimal stopping techniques and uses the fact that $\mathcal{V}$ is the pointwise minimum of $\mathbb{H}$.

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(3) Then $\mathfrak{V} \leq \mathcal{V} \leq \mathbb{V}$. Now apply viscosity comparison (if it holds) so that $\mathbb{V} \leq \mathfrak{V}$ and hence

$$\mathcal{V} = \mathbb{V} = \mathfrak{V}$$

is **continuous**, the **unique viscosity solution** of the QVIs, and the **pointwise infimum** of $\mathbb{H}$.

Discussion of the Approach

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Challenges when applying the approach:

- The bottleneck is viscosity comparison, needed for continuity of $\mathcal{V}$;
- Admissibility of the candidate optimal control has to be checked on a case-by-case basis. This is a general problem though.

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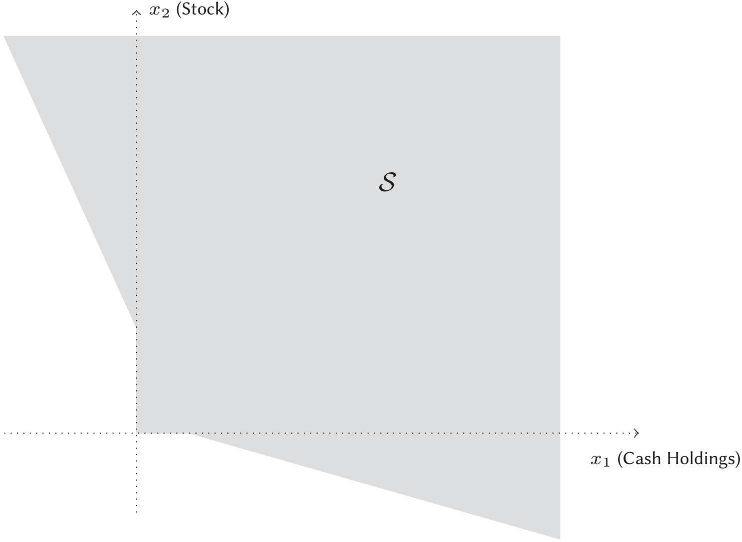
A portfolio $x \in \mathbb{R}^2$ is **solvent** if it has a positive liquidation value $\mathrm{L}(x) \geq 0$, where

$$\mathrm{L}(x) \triangleq \begin{cases} x_1 + x_2 - C(-x_2) & \text{if } x_2 < 0,\\ x_1 + \big(x_2 - C(-x_2)\big)^+ & \text{otherwise.} \end{cases}$$

The set $\mathcal{S} \subset \mathbb{R}^2$ of solvent portfolios is called the **solvency region**.

The Solvency Region for Constant + Proportional Costs



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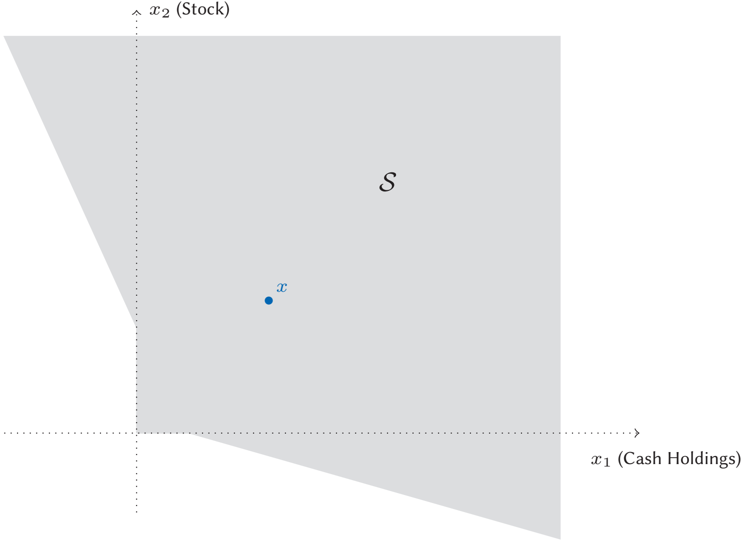

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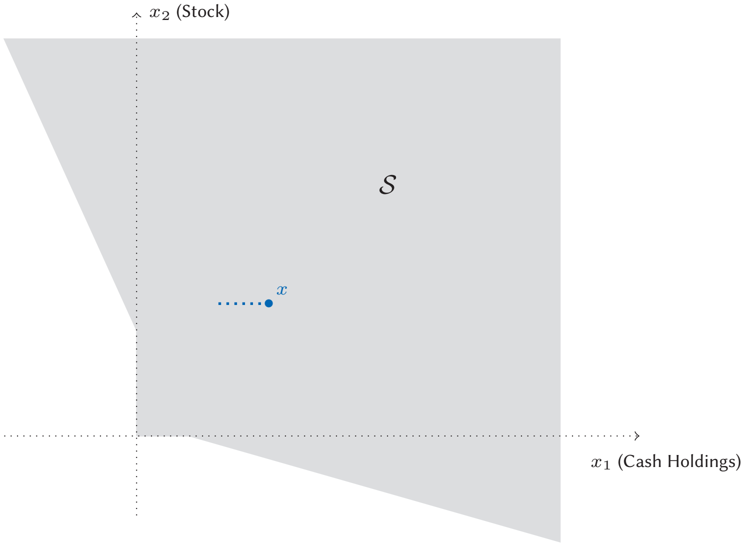

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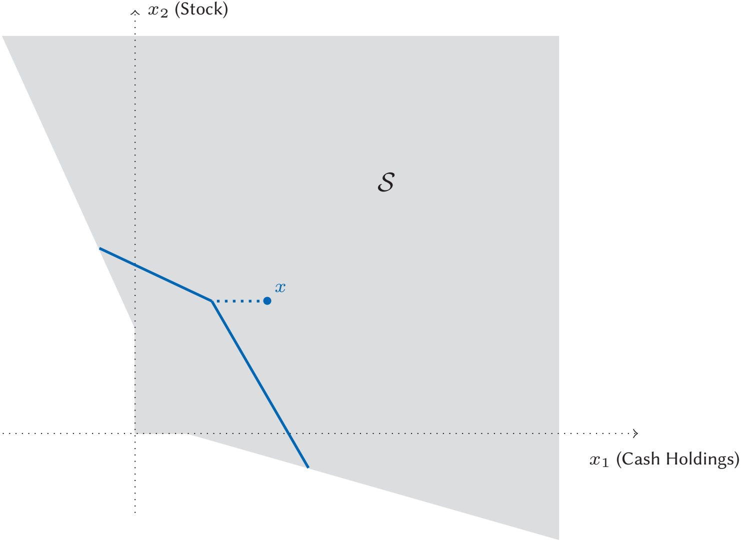

The Optimization Criterion

For simplicity, we restrict to (positive) **power utility**

$$U : \mathbb{R}_+ \to \mathbb{R}, \qquad \ell \mapsto U(\ell) \triangleq \frac{1}{p}\ell^p \qquad \text{with } p \in (0,1).$$

The objective is to **maximize expected utility of terminal wealth**, i.e.

$$\mathcal{V}(t,x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X^{\Lambda}_{t,x}(T)\big)\Big)\Big],$$

where $\mathcal{A}(t,x)$ denotes the set of **admissible strategies** Λ for the initial state (t,x), i.e. the set of strategies Λ for which

$$\mathrm{L}\big(X^{\Lambda}_{t,x}\big) \geq 0 \qquad \text{on } [t,T].$$

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The general theory **does not apply directly** due to, e.g., the constrained state space. However, this is to be expected. It should not be too difficult to adapt the arguments.

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Idea: Localize the viscosity argument by splitting the solvency region as follows:

- $x_1 \geq 0$ and $x_2 \geq 0$: **Long Portfolios**,
- $x_1 \geq 0$ and $x_2 < 0$: **Short Portfolios**,
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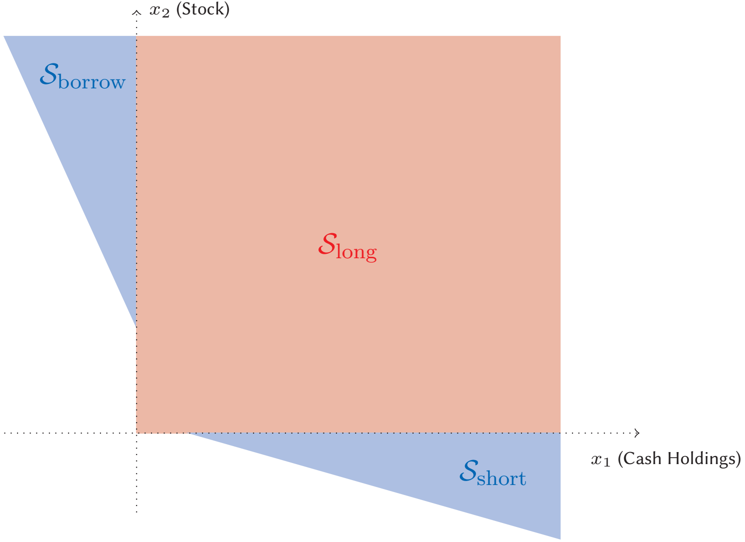
Difficulty: The QVIs have a **non-local** term: $\mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)$.

Splitting the Solvency Region



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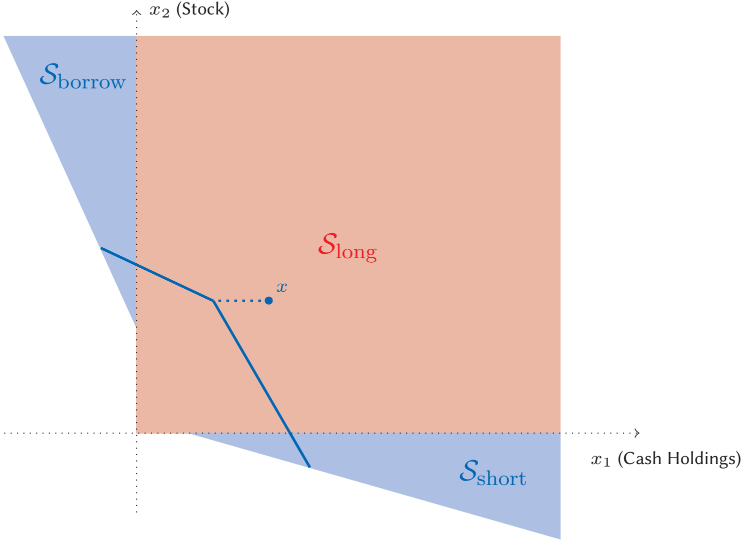

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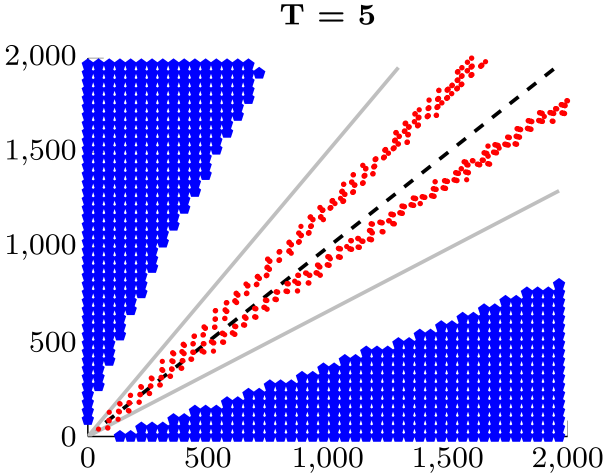
(3) The candidate **optimal strategy** does not accumulate and is indeed optimal.

Numerical Results: Constant + Proportional Costs

Numerical Example: Constant + Proportional Costs



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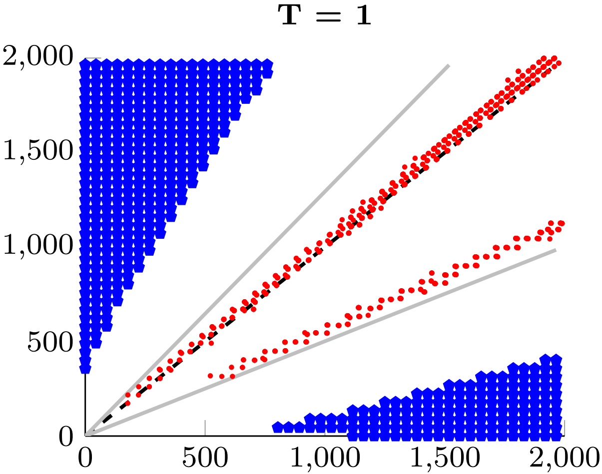

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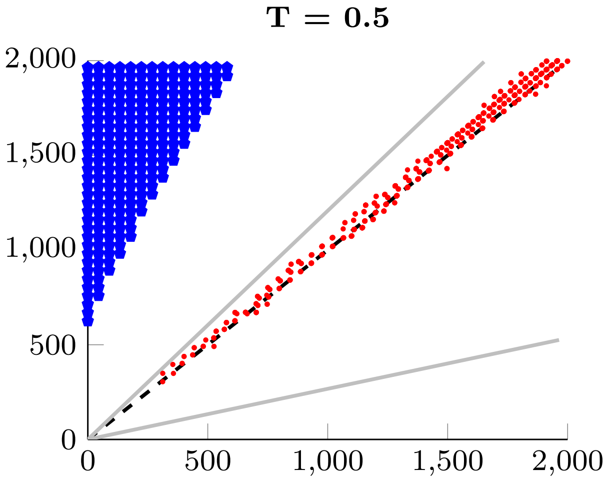

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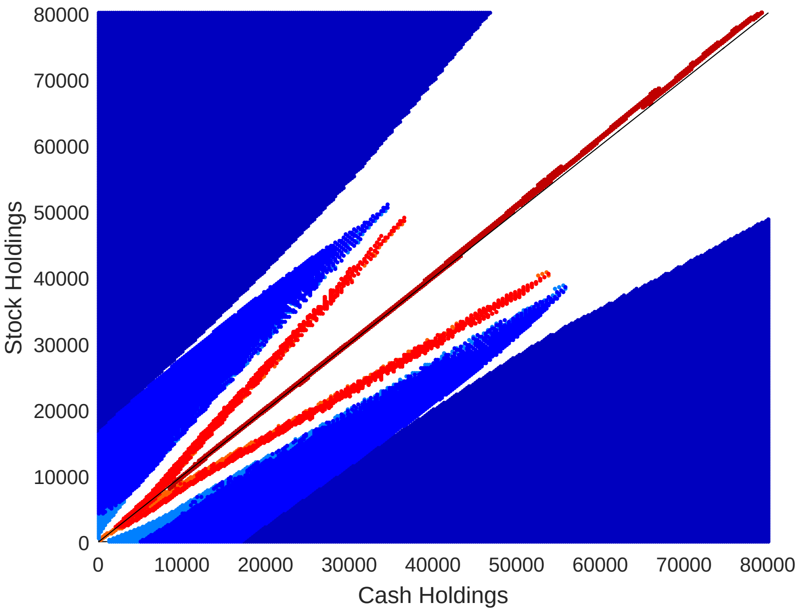

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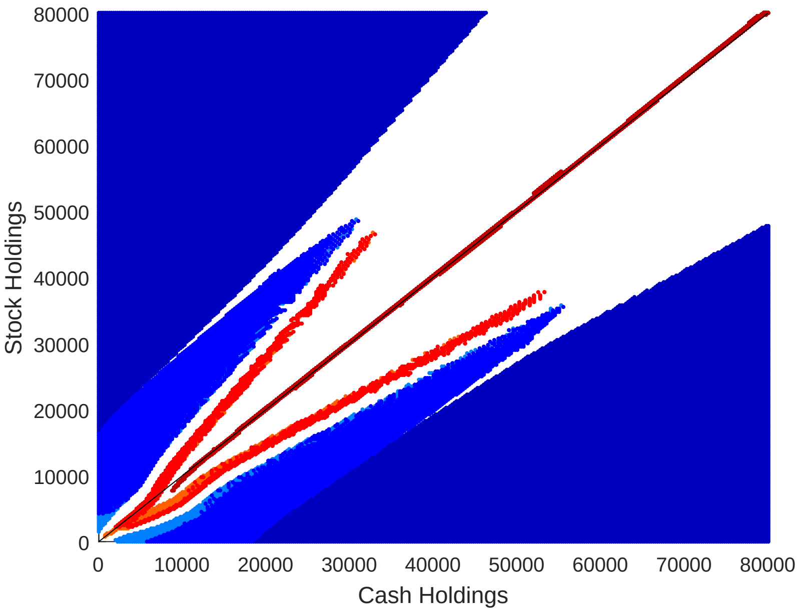

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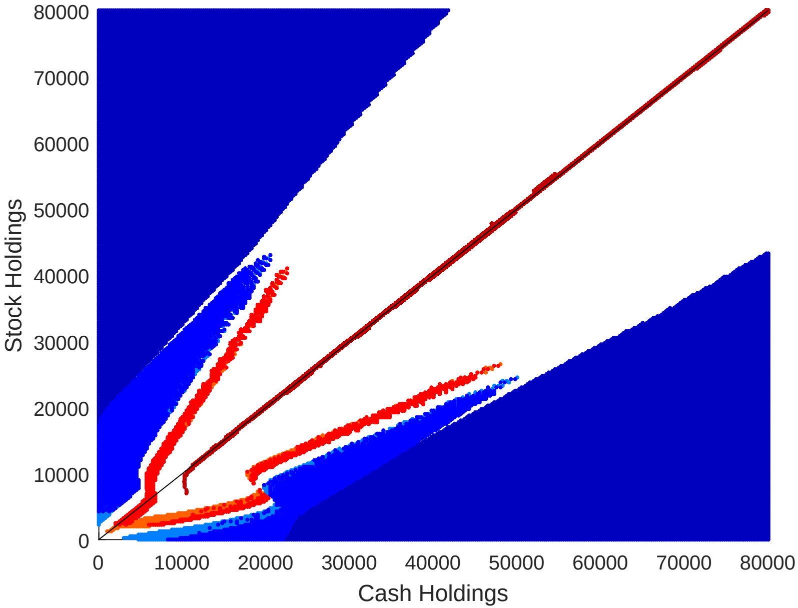

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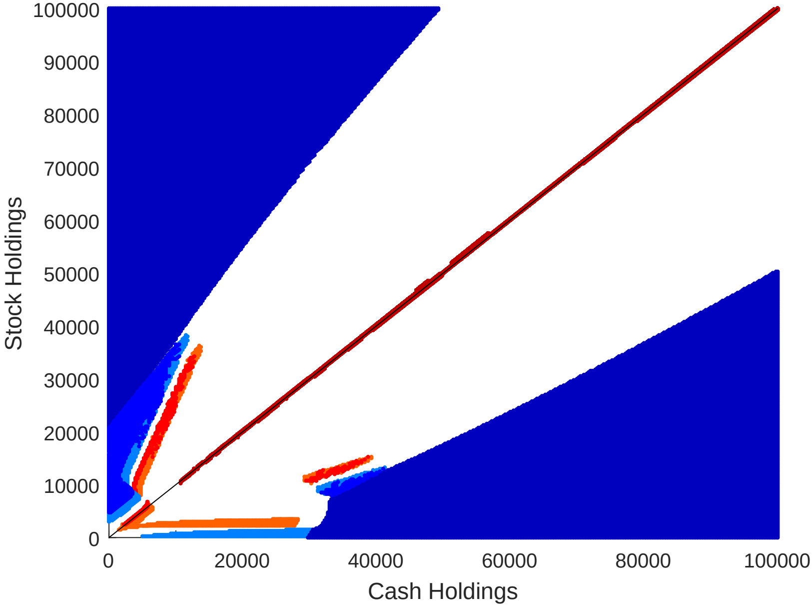

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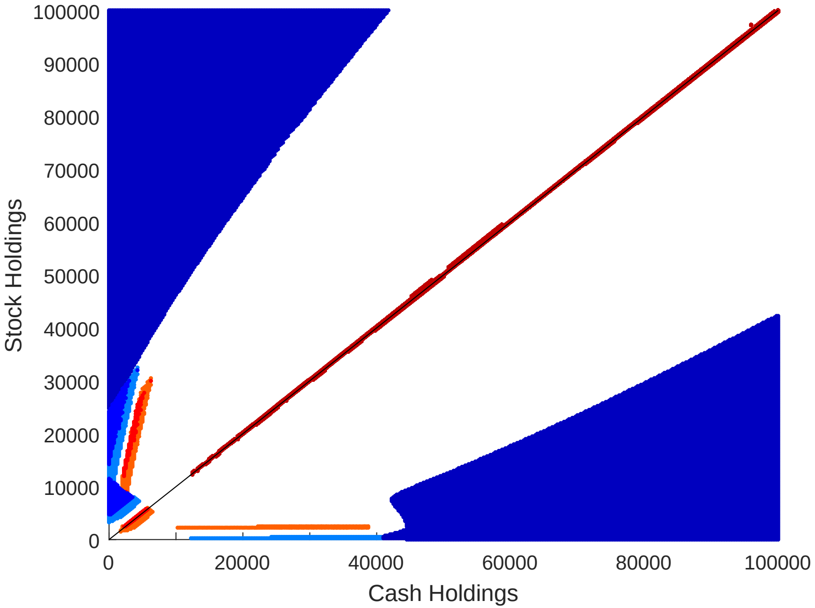

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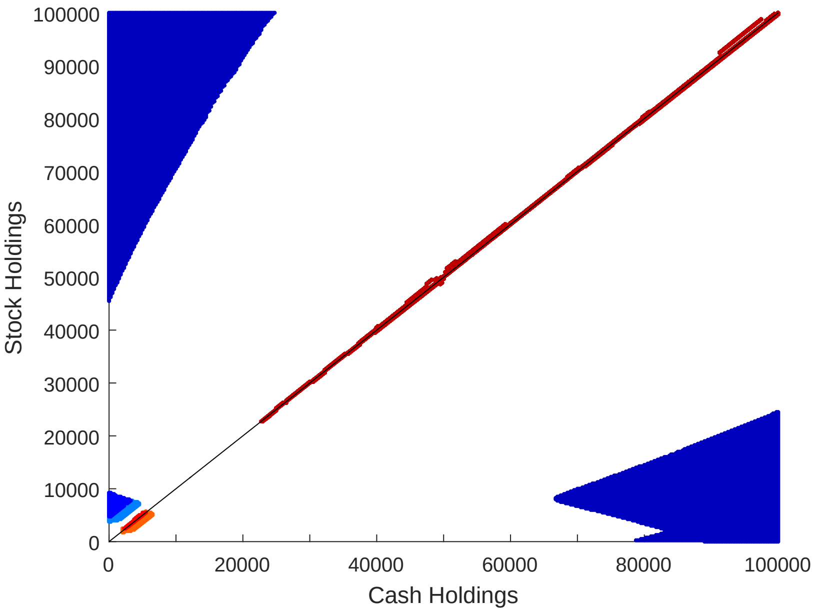

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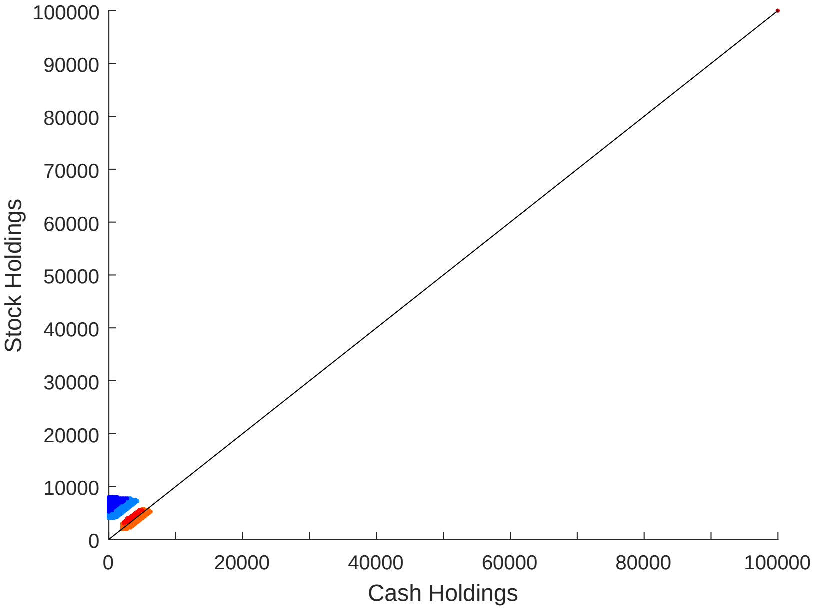

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Thank you for the attention!