Multivariate Newton & Lagrange Interpolation

Michael Hecht [1]

MOSAIC Group, Center for Systems Biology Dresden,

Max Planck Institute of Molecular Cell Biology and Genetics, Dresden, Germany

In scientific computing, the problem of interpolating a multivariate function $f : \Omega \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$, $m \in \mathbb{N}$, is ubiquitous. It occurs for instance in subroutines of *numerical ODE & PDE solvers, multidimensional regression, optimization problems, quadrature computations* and analysis of *BIG DATA*. Because of their simple differentiation and integration, as well as their pleasant vector space structure, real polynomials $Q \in \Pi_{m,n}$ in m variables of degree $\deg(Q) \leq n$, $m, n \in \mathbb{N}$ possessing $N = \binom{m+n}{n}$ coefficients, are a standard choice as interpolants. Since the 18^{th} century the classical $1D$– *Newton & Lagrange Interpolation* and their numerical solvers as the 1D–*divided difference scheme (DDS)*, provide a fast and numerical robust approximation scheme for all *Sobolev functions* $f \in H^1((a,b), \mathbb{R})\,, a < b \in \mathbb{R}$. However, no multi–dimensional generalization of these methods has been established so far.

Alternative schemes as *spline & wavelet interpolation* or *fast Fourier transformation (FFT)* can not overcome the *curse of dimensionality*, i.e., scale exponentially in $\mathcal{O}(r^m)$ with dimension $m \in \mathbb{N}$ depending on the resolution $r \in \mathbb{R}^+$. *Machine learning approaches* can deal with high dimensions but loose any control of their approximation abilities, i.e., the distance of the learned model to the ground trough can usually not be estimated. Further, the thereby provided representation of the model as a weighted artificial neural network is rather hard to understand.

Recently, we used concepts of *algebraic geometry* to give a feasible notion of *unisolvent nodes* $P_{m,n} \subseteq \Omega = [-1,1]^m$. That is: there is exactly one polynomial $Q_{m,n,f}$ of degree n that can fit a given function f on $P_{m,n} \subseteq \Omega$, i.e., $Q_{m,n,f}(p) = f(p)\,, \forall\, p \in P_{m,n}$.

In light of this fact, we generalised the 1D Newton and 1D (barycentrical) Lagrange Interpolation schemes to arbitrary dimension yielding algorithms with runtime and storage complexities $\mathcal{O}(N^2)$, $\mathcal{O}(N)$ and $\mathcal{O}((m+n)N$, $\mathcal{O}(N)$, respectively. Since $N \in \mathcal{O}(m^n)$ and the 1D approximation estimates translate naturally to multi–dimensions, for the first time, we propose interpolation schemes that can overcome the curse of dimensionality, i.e., scale polynomial in $\mathcal{O}(m^{n+1})$ for fixed degree $n \in \mathbb{N}$ and uniformly approximate all Sobolev functions $f \in H^k(\Omega, \mathbb{R})\,, k > m/2$ for $n \to \infty$.

Related Publications

M. Hecht, K. B. Hoffmann, B. L. Cheeseman, and I. F. Sbalzarini, *Multivariate Newton interpolation*, *arXiv preprint arXiv:1812.04256.*

Hecht, M., and Sbalzarini, I. F. Fast interpolation and Fourier transform in high-dimensional spaces. In *Intelligent Computing. Proc. 2018 IEEE Computing Conf., Vol. 2,* (London, UK, 2018), K. Arai, S. Kapoor, and R. Bhatia, Eds., vol. 857 of *Advances in Intelligent Systems and Computing*, Springer Nature, pp. 53–75.

[1]Email: hecht@mpi-cbg.de