Term Structure Modelling from the SOFR Perspective

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Outline

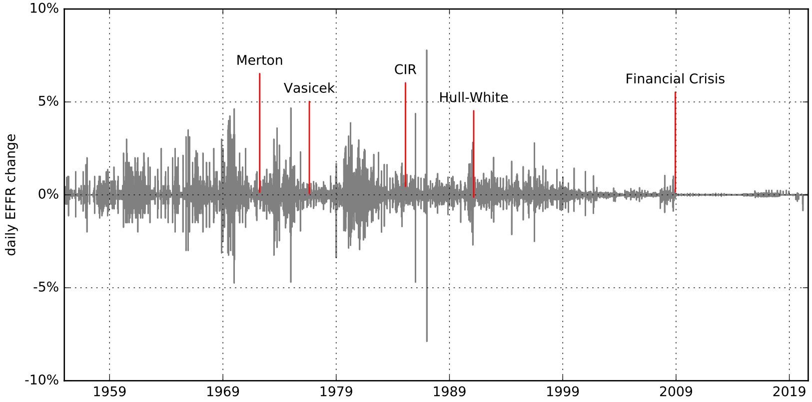
Interest rate term structure modelling

- Historically, interest rate term structure modelling has been based on rates of substantially longer time to maturity than overnight:
  - directly as in the LIBOR Market Model
  - or indirectly, in the sense that even models based on continuously compounded short rate are typically calibrated to term rates of longer maturities
- Any regard to a market overnight rate traditionally has been at best an afterthought
- However, with SOFR this situation is reversed:
  - The overnight rate now is the primary market observable,
  - and term rates are less readily available
  - and therefore must be inferred (for example from futures)

Implications for modelling

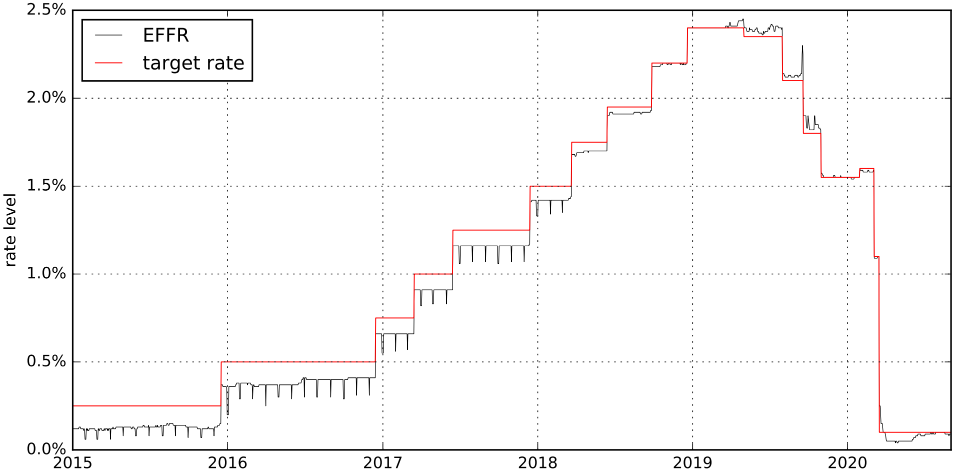
- Idiosyncracies of the overnight rate cannot be ignored when constructing interest rate term structure models in a SOFR-based world
- and more than longer term rates, these idiosyncracies are driven by monetary and regulatory policy
  - Piazzesi (2005) Grzelak et al. (2008)
- In particular, the model must reflect jumps with deterministic jump times, referred to in literature as stochastic discontinuities, see e.g.
  - Fontana et al. (2020)
  - Keller–Ressel et al. (2018)
  - Kim and Wright (2014)

Short rate history

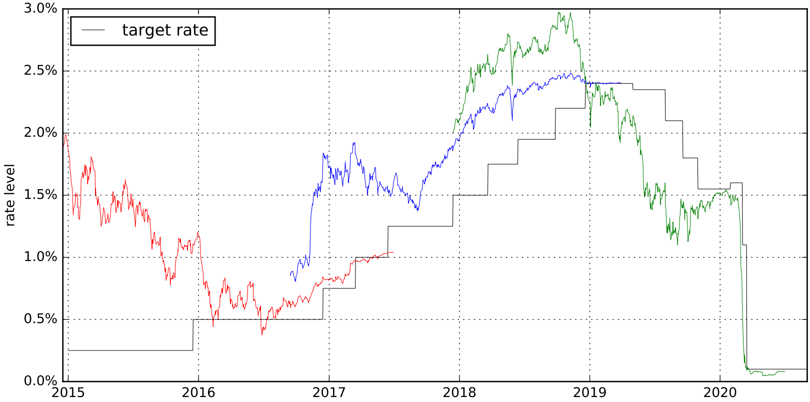


Empirical daily EFFR changes and the history of short rate models

EFFR vs FOMC target rate history



Target rate vs. forward rates

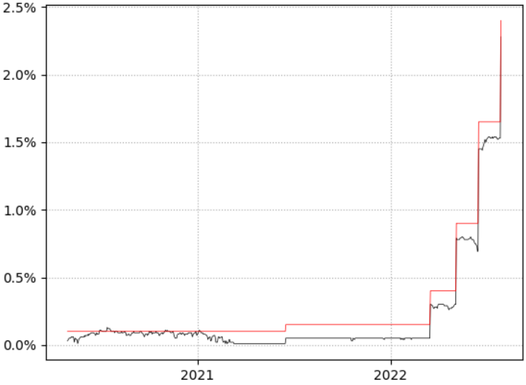


Target rate vs. forward rates implied by three different 30-day Fed Funds futures

Secured Overnight Funding Rate

SOFR and FOMC target rate history 2015-2020

Secured Overnight Funding Rate



SOFR and FOMC target rate history 2021-2022

Notable features of SOFR dynamics

- EFFR and SOFR both appear to follow similar (largely) stepwise paths, suggesting that the Fed Funds target rate plays an important role in determining SOFR dynamics as well.
- Prominent spikes in SOFR, often coinciding with month end dates, have mostly disappeared since the end of 2020.
- The spread to the FOMC target rate is relatively constant.

Outline

Forward rate dynamics

- The main empirical feature of the target rate is that it is piecewise flat between the FOMC meeting dates at which a policy change has occurred.
- Seek to reconcile short rate dynamics determined primarily by jumps at known times with forward rates following primarily diffusive dynamics.
- An interpretation of the forward rates deduced from Fed Fund futures is that they reflect the expectations of prospective FOMC target rate changes. The diffusive dynamics of forward rates then reflect the changes in those expectations.

Gaussian forward rate model with piecewise volatility

$$f(t,T) = f(0,T) + \alpha(t,T) + \sum_{i=1}^{n} \int_0^t \xi_i(s,T) dZ_i(s)$$

where $f(0,T)$ is the initial term structure, $\alpha(t,T)$ a deterministic drift and $dZ_i(s)$ the Wiener increment corresponding to the i_{th} FOMC date with correlation $dZ_i(t)dZ_j(t) = \rho_{i,j}dt$. The volatility term is defined as follows:

$$\xi_i(t,T) = \xi_i \mathbb{1}(t < x_i)\mathbb{1}(T \geq x_i)$$

where x_i denotes the i^{th} FOMC meeting date. Solving the integral:

$$f(t,T) = f(0,T) + \alpha(t,T) + \sum_{i=1}^{n} \xi_i \mathbb{1}(T \geq x_i) Z_i(t \wedge x_i)$$

Forward rate dynamics intuition

- Each stochastic component corresponds to an FOMC date x_i and any changes to the target rate are carried forward from that date
- The indicator function $\mathbb{1}(T \geq x_i)$ ensures that the i^{th} factor is only applied to forwards with maturities greater or equal to x_i
- No diffusion for forward rates with maturities prior to the first (or next) FOMC meeting date.

Piecewise constant short rates

These forward rate dynamics create the piecewise constant dynamic in the short rate, setting $r(t) = f(t, t)$:

$$r(t) = f(t,t) = f(0,t) + \alpha(t,t) + \sum_{i=1}^{n} \int_{0}^{t} \xi_i(s,t) dZ_i(s)$$

Solving the integral:

$$r(t) = f(0,t) + \alpha(t,t) + \sum_{i=1}^{n} \xi_i \mathbb{1}(t \geq x_i) Z_i(x_i)$$

From this it is evident that the short rate has no diffusion up until the first FOMC date, at which point it picks up all the diffusion from the forward rate accumulated up until this point in time.

Transformation to independent components

Transformation to independent components makes the model consistent with the HJM framework, thus facilitating derivation of risk–neutral dynamics.

Define Σ to be the covariance matrix of the vector $dZ = [dZ_1, ..., dZ_n]$. There exists a transformation matrix γ, such that $\Sigma = \gamma\gamma^T$ and $dZ = \gamma dW$, to yield uncorrelated Wiener increments $dW = [dW_1, ..., dW_n]$.

Therefore:

$$dZ_i = \sum_{j=1}^{n} \gamma_{i,j} dW_j$$

Forward rates under risk neutral measure

Rewrite the forward rate dynamics:

$$\sum_{i=1}^{n}\int_0^t \xi_i(s,T)dZ_i(s)=\sum_{j=1}^{n}\int_0^t \sigma_j(s,T)dW_j(s)$$

where

$$\sigma_j(t,T)=\sum_{i=1}^{n}\xi_i\gamma_{i,j}\mathbb{1}(t<x_i)\mathbb{1}(T\geq x_i)$$

Therefore:

$$\begin{aligned}f(t,T)=f(0,T)&+\sum_{j=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n}\xi_q\xi_i\gamma_{q,j}\gamma_{i,j}\mathbb{1}(T\geq x_{q\vee i})(T-x_i)[t\wedge x_q\wedge x_i]\\&+\sum_{j=1}^{n}\sum_{i=1}^{n}\xi_i\gamma_{i,j}\mathbb{1}(T\geq x_i)W_j(t\wedge x_i)\end{aligned}$$

Short rates under risk neutral measure

Similarly:

$$r(t) = \underbrace{f(0,t) + \sum_{j=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n}\xi_q\xi_i\gamma_{q,j}\gamma_{i,j}\mathbb{1}(t \geq x_{q\vee i})(t - x_i)[x_q \wedge x_i]}_{\text{deterministic term (**)}}$$

$$+ \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{n}\xi_i\gamma_{i,j}\mathbb{1}(t \geq x_i)W_j(x_i)}_{\text{stochastic term (*)}}$$

Fed Funds and 1M SOFR futures

Contracts based on the arithmetic average of the EFFR or SOFR over a specified contract month with terminal value:

$$\widetilde{F}_m(\tau_{m,n_m}) = 100 - R_m = 100\left(1 - \frac{1}{n_m}\sum_{i=1}^{n_m} r(\tau_{m,i})\right)$$

$\widetilde{F}_m(t)$: the futures contract index for reference month m at time t
m : the number of months from the current month
$\tau_{m,i}$: the date corresponding to day i in month m
n_m : the total days in month m.
The expected value at t under the spot risk neutral measure is:

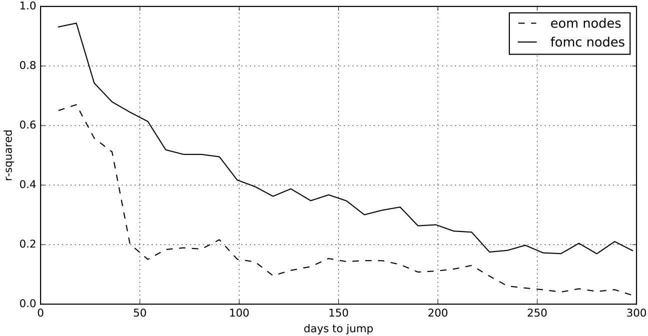
$$F_m(t) = E_t[\widetilde{F}_m(\tau_{m,n_m})] = 100\left(1 - \tfrac{1}{n_0}\left(\sum_{i=1}^{n_0} 1_{(t>\tau_{0,i})} r(\tau_{0,i}) + \sum_{i=1}^{n_0} 1_{(t\leq\tau_{0,i})} E_t[r(\tau_{0,i})]\right)\right)$$

Calibration assumptions

- Use first 12 monthly Fed Funds futures contracts
- Piecewise flat initial term structure
- Drift impact is negligible
- No spikes
- Constant EFFR to target rate spread
- The calibration error $e_m(t)$ for month m: $e_m(t) = (|F_m(t) - \widetilde{F}_m(t)| - h_m)^+$ with $h_0 = 0.0025$ and $h_m = 0.005$ for $m \neq 0$ (accounts for bid/offer)
- The assumptions amount to persistent piecewise flat forward rate term structure

The calibration is performed on daily data in the period from January 2015 to September 2020

Futures as a predictor of policy rate changes



R-squared of EFFR realised vs forward rates for different forward periods

Time homogeneity for estimation

In the original model the volatility was defined as function of specific FOMC dates. To facilitate empirical estimation, redefine the volatility as a function of the number of meeting dates between t and T:

$$\xi_i(t,T) = \xi_i \mathbb{1}(i \leq \mathcal{A}_{t,T})$$

where $\mathcal{A}_{t,T}$ reflects the number of meeting dates between t and T:

$$\mathcal{A}_{t,T} := \Big|\{x_1, ..., x_m | t < x_i \leq T\}\Big|$$

where x_i denotes the i^{th} FOMC meeting date. The intuition behind this construction is that each stochastic component corresponds to evolving expectations related to an FOMC policy decision based on its order from the current state time.

3M SOFR futures

Contracts based on compounding of SOFR over a reference period with terminal value:

$$\widetilde{F}_q^{s3}(\tau_{q,n_q}) = 100 - 100 \times \frac{360}{n_q}\left[\prod_{i=1}^{n_q}\left\{1_{(\tau_{q,i}\in \mathrm{b})}\left(1+\frac{d_i r_s(\tau_{q,i})}{360}\right)\right\} - 1\right]$$

q : the number of IMM quarters from the current trading quarter

$\tau^*_{q,i}$: the date corresponding to day i in quarter q

n_q : denoting the total days in quarter q.

b : set of US government securities business days

d_i: the number of days the rate $r_s(\tau_{q,i})$ applies.

3M SOFR futures price

The expected value at t under the spot risk neutral measure can be approximated as:

$$F_0^{s3}(t) = E_t[\widetilde{F}_q^{s3}(\tau_{0,n_0})] \approx 100\left(1 - \frac{360}{n_0}\left[\prod_{i=1}^{n_0}\left\{1_{(\tau_{0,i}\in \mathrm{b})}\left(1 + \frac{d_i r_s^*(\tau_{0,i})}{360}\right)\right\} - 1\right]\right)$$

where

$$r_s^*(\tau_{0,i}) = 1_{(t>\tau_{0,i})} r_s(\tau_{0,i}) + 1_{(t\leq\tau_{0,i})} E_t[r_s(\tau_{0,i})]$$

We verified by Monte Carlo simulation that this approximation introduces an error of 0.5 bp or less in the futures rate.

Empirical estimation

For each day in the observation period assume forward rates are piecewise constant between FOMC dates and solve:

$$\mathrm{f}(t_a) = \arg\min_{\mathrm{f}(t_a)} O(t_a)$$

where $\mathrm{f}(t_a)$ is the vector of forward rates $f(t_a, t_i)$ which are piecewise constant between FOMC dates. The objective function is defined as the sum of square errors between the price given $\mathrm{f}(t_a)$ and the market price for monthly and quarterly futures:

$$O(t_a) = \sum_j (\hat{F}_j^1(t_a) - F_j^1(t_a))^2 + \sum_j (\hat{F}_j^3(t_a) - F_j^3(t_a))^2$$

Empirical estimation

Using principal component analysis on the calibrated states we can estimate:

$$\Delta f^*(t_a, T) = f^*(t_a, T) - f^*(t_{a-1}, T) = \sum_{j=1}^{\beta} \sum_{i=1}^{n} w_{i,j} \mathbb{1}(i \leq \mathcal{A}_{t_a,T}) s_j(t_a)$$

The above equation connects the empirical results to the model as follows. Let $\Delta W_j(t_a) = W_j(t_a) - W_j(t_{a-1})$, we can write:

$$\Delta f(t_a, T) = \sum_{j=1}^{n} \sum_{i=1}^{n} \gamma_{i,j} \mathbb{1}(i \leq \mathcal{A}_{t_a,T}) \sigma_j \Delta W_j(t_a) \qquad (1)$$

The empirical results are connected to the model with $w_{i,j} = \gamma_{i,j}$ and $s_j(t_a) = \sigma_j \Delta W_j(t_a)$.

Fitted model: RMSE

instrument	M0	M1	M2	M3	M4	M5	M6	Q0	Q1	Q2	Q3	Q4
1 factor	1.2	1.7	6.6	12.8	19.6	26.0	30.7	7.2	22.9	36.3	47.4	68.6
2 factor	1.1	1.4	1.5	1.4	2.2	2.6	2.6	0.8	2.1	1.9	5.0	10.6
3 factor	1.1	1.4	1.5	1.3	1.6	1.5	1.3	0.8	1.0	1.0	1.1	2.1
Skov & Skovmand	2.9	3.1	3.3	2.6	2.1	1.6	1.6	0.9	1.3	1.8	0.9	1.8

Figure: Root mean squared error (RMSE) for the model on the same data set as used by Skov, J. B., & Skovmand, D. (2021). Dynamic term structure models for SOFR futures. *Journal of Futures Markets*, 41, 1520–1544.

Factor term structure

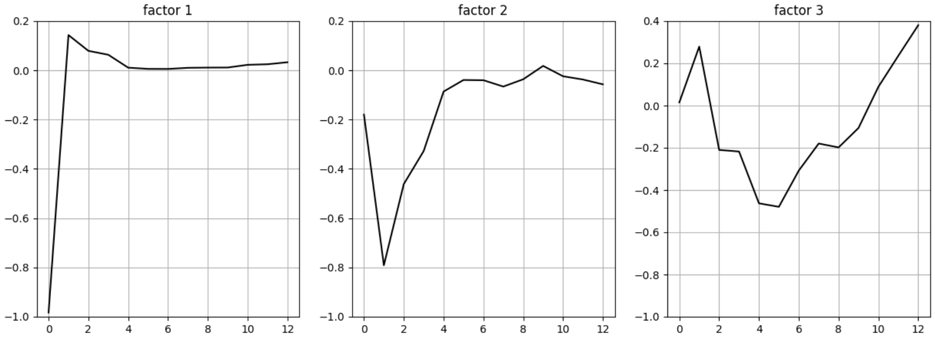


Figure: γ vectors for the first three factors

Factor states empirical dynamics

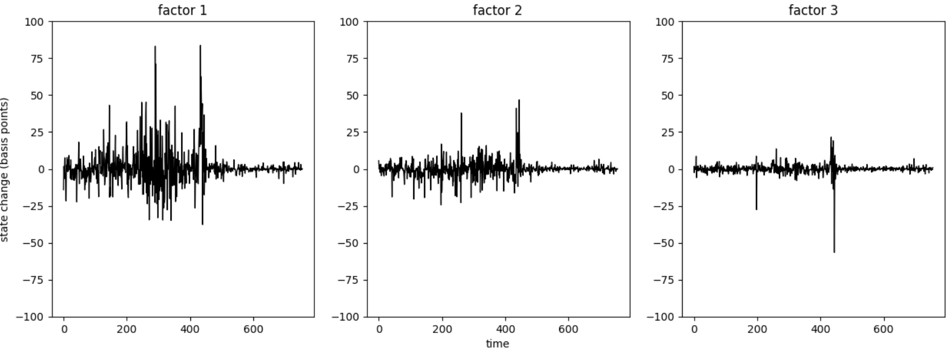


Figure: estimated factor state changes

Factor states empirical dynamics

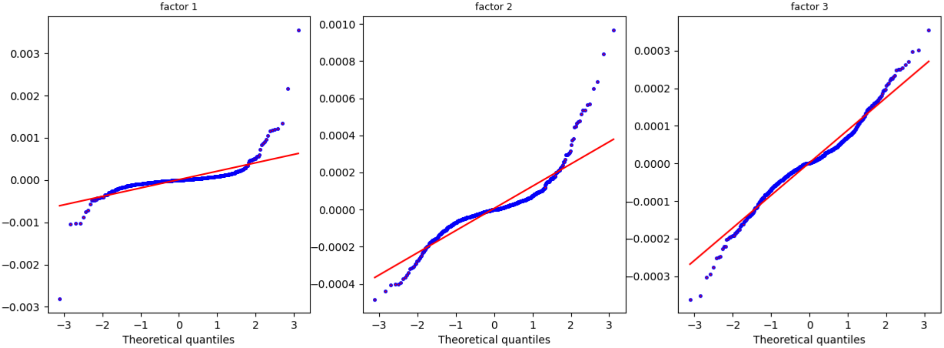


Figure: empirical factor states quantile quantile plots

A Heston/Hull–White model

Start with a 1-factor quasi-Gaussian model (QG1) model with HJM defined volatility:

$$\sigma(t,T) = \chi(t)\phi(T)$$

where $\chi(t)$ is generally stochastic.

$$df(t,T) = F(t,T)dt + \sigma(t,T)dW(t)$$

where

$$F(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

$$\implies f(t,T) - f(0,T) = \int_0^t F(s,T)ds + \phi(T)\int_0^t \chi(s)dW(s)$$

A Heston/Hull–White model

Set $T = t$ and differentiate with respect to t to express the spot $r(t)$ in the form

$$r(t) - f(0,t) = z(t) = \int_0^t F(s,t)ds + \phi(t)\int_0^t \chi(s)dW(s), z(0) = 0$$

Define:

$$\phi(T) = \exp\left(-\int_0^T \lambda(v)dv\right) \text{ and } \chi(t) = \sigma(t)\exp\left(\int_0^t \lambda(v)dv\right)$$

A Heston/Hull–White model

The SDE becomes:

$$dz(t) = \left\{ \frac{d}{dt}\left[\int_0^t F(s,t)ds \right] + \lambda(t) \int_0^t F(s,t)ds \right\} dt - \lambda(t)z(t)dt + \sigma(t)dW(t)$$

in which the part of the drift term involving $F(t,T)$ simplifies to

$$\Phi(t) = F(t,t) + \int_0^t \frac{\partial}{\partial T} F(s,T)\bigg|_{T=t} ds + \lambda(t) \int_0^t F(s,t)ds$$

$$= \int_0^t \sigma^2(s) \exp\left(-2 \int_s^t \lambda(v)dv \right) ds = \int_0^t \sigma^2(s,t)ds$$

A Heston/Hull–White model

The volatility $\sigma(.)$ is made stochastic by incorporating a Heston process $v(.)$ in it:

$$\sigma(t) \to \sigma(t)\sqrt{v(t)}$$

which constitutes an affine system under the HJM spot measure:

$$\begin{aligned} dz(t) &= [\Phi(t) - \lambda(t)z(t)]dt + \sigma(t)\sqrt{v(t)}dW(t), z(0) = 0 \\ d\Phi(t) &= [\sigma^2(t)v(t) - 2\lambda(t)\Phi(t)]dt, \Phi(0) = 0 \\ dv(t) &= \theta(t)(1 - v(t))dt + \alpha(t)\sqrt{v(t)}dU(t), v(0) = 1 \end{aligned}$$

with

$$\langle dW(.), dU(.)\rangle(t) = \rho dt$$

Piecewise Heston/Hull–White

- Each factor in the piecewise HJM model is extended in the same manner as HW in the HHW model.
- This model inherits the piecewise continuous structure, but features stochastic volatility.
- This set–up provides ample flexibility to calibrate to a volatility term structure (since each factor uniquely impacts different aspects of the forward rate term structure) as well as an option–implied volatility skew and smile across different expiries.
- The level of flexibility is regulated with the choice of the number of factors and time dependence of the HHW variables.

Piecewise Heston/Hull–White

Define the HJM volatility as follows:

$$\sigma_j(t,T)=\sum_{i=1}^{n}\mathbb{I}_{\{i\leq\mathcal{A}(t,T)\}}\chi_j(t)\phi_j(T)\gamma_{i,j}$$

where

$$\phi_j(T)=\exp\left(-\int_0^T\lambda_j(s)ds\right)$$

$$\chi_j(t)=\sigma_j(t)\sqrt{v_j(t)}\exp\left(\int_0^t\lambda_j(s)ds\right)$$

$v(t)$ evolves with a Heston dynamic:

$$dv(t)=\theta(t)(1-v(t))dt+\alpha(t)\sqrt{v(t)}dU(t),\,v(0)=1$$

with

$$\langle dW_j(.),dU_j(.)\rangle(t)=\rho_j dt \text{ and } \langle dW_i(.),dU_j(.)\rangle(t)=0,\;\text{for } i\neq j$$

Outline

1. From SOFR via Fed Funds to Term Structure Modelling
   - Changing interest rate benchmarks
   - Effective Fed Funds rate (EFFR)
   - Features of the SOFR dynamics
2. Modelling
   - Target Rate Step Model
   - Time homogeneous Target Rate Step Model
   - Introducing Stochastic Volatility
3. Model Features
   - Calibration to Options
   - Accrual period behaviour
   - Mean reversion

Pricing Options on Futures

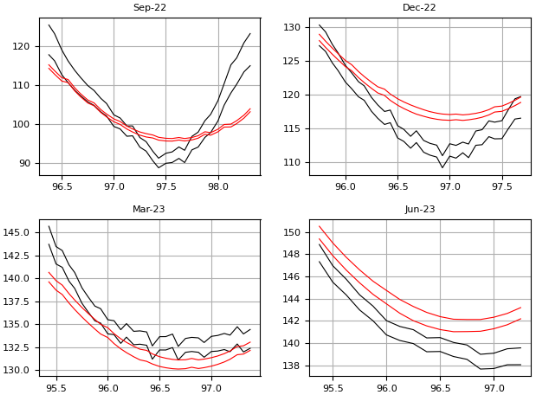
The value of a call option at time t, expiring at T_e with strike K on the futures contract under the spot risk neutral measure:

$$C(t, T_e, F(T_i, T_k), K) = E_\beta\left[\frac{1}{\beta(T_e)}(F(T_i, T_k) - K)^+ | \mathcal{F}_t\right]$$

Options on SOFR futures are specified with American–style exercise. However, we use them to approximate European–style implied volatilities, as is common in practice.

Calibration to options

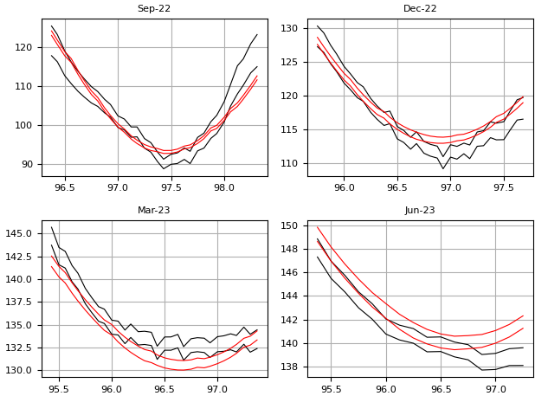
- Calibrate to first four options on 3M SOFR futures
- Initially set constant model parameters



Model prices at 5% simulation CI (red) vs market bid/offer (black)

Time dependent parameters

- Time–dependent piecewise constant stoch. volatility parameters.
- Each piecewise section corresponds to each option expiry.

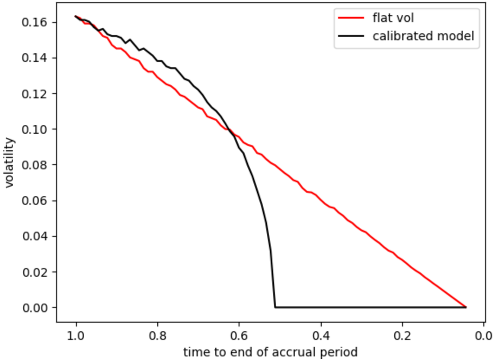


Accrual period

- A prevalent approach in the LIBOR to SOFR transition, as reflected in literature (see Lyashenko and Mercurio (2019)), is the adaptation of existing LIBOR based modelling to SOFR.
- A highly practical problem stemming from this approach is the behaviour of options in the accrual period of the SOFR term rate.
- Existing LIBOR–based pricing models require an artificially induced decay of the “SOFR term rate” volatility within its accrual period.

Accrual period

- Setting a constant sigma and removing the indicator functions in the volatility function results in a linearly decaying implied volatility, consistent with Lyashenko and Mercurio (2019).
- The fully specified model indicates a decay to zero until the next FOMC date



Mean reversion

- Mean reversion is embedded in the definition of σ

$$\sigma_j(t,T) = \sigma_j(t)\sqrt{v_j(t)}\exp\Big(-\int_t^T \lambda_j(s)ds\Big)\sum_{i=1}^n \mathbb{I}_{\{i\leq\mathcal{A}(t,T)\}}\gamma_{i,j}$$

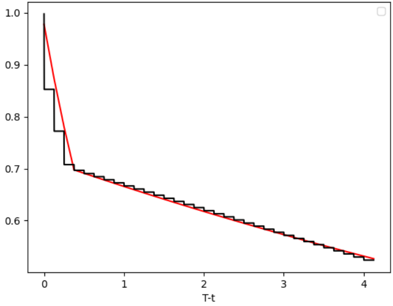
- The term $\exp\Big(-\int_t^T \lambda_j(s)ds\Big)$ decays volatility as a function of $(T-t)$ which results in short rate mean reversion
- Through an appropriate choice of γ it is possible to mimic the mean reversion term

$$\sum_{i=1}^n \mathbb{I}_{\{i\leq\mathcal{A}(t,T)\}}\gamma_{i,j} \approx \exp\Big(-\int_t^T \lambda_j(s)ds\Big)$$

Mean reversion

- When the model is estimated the γ corresponding to the first factor indeed resembles mean reversion
- λ can be defined to reflect γ:

$$\lambda_j(s) = \begin{cases} 0.9, & s - t < 0.5 \\ 0.08, & \text{otherwise} \end{cases}$$



Mean reversion

- The model gives a monetary policy foundation to the presence of mean reversion.
- Mean reversion is linked to FOMC policy rates, i.e. monetary policy decisions.
- The Federal Reserve acts to mean revert rates in its management of the economic cycle.
- The market expects this, which through our model, is detectable in the behaviour of futures prices.

Papers

This work is in the process of being written up in three papers:

- Gellert, Karol and Schlögl, Erik, Short Rate Dynamics: A Fed Funds and SOFR perspective (January 11, 2021). Available at SSRN: `https://ssrn.com/abstract=3763589` or `http://dx.doi.org/10.2139/ssrn.3763589`
- Gellert, Karol and Schlögl, Erik, Term Structure Modelling from the SOFR Perspective (working paper in preparation)
- Brace, Alan, Gellert, Karol and Schlögl, Erik, SOFR Term Structure Dynamics — Discontinuous Short Rates and Stochastic Volatility Forward Rates (working paper in preparation)