Multiobjective Replacements for Set Optimization and Robust Multiobjective Optimization (Slides Part I)

joint work with Ernest Quintana and Stefan Rocktäschel

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What is Set Optimization?

- Minimizing a **scalar-valued** objective function $f\colon \Omega \to \mathbb{R}$ over a non-empty set Ω:

$$\bar{x} \in \Omega \text{ minimal solution } \Leftrightarrow f(\bar{x}) \leq f(x) \text{ for all } x \in \Omega .$$

- **Multiobjective optimization:** Minimizing a **vector-valued** objective function

$$f\colon \Omega \to \mathbb{R}^m$$

over a non-empty set Ω:

$$\bar{x} \in \Omega \text{ efficient solution } \Leftrightarrow (\{f(\bar{x})\} - \mathbb{R}^m_+) \cap f(\Omega) = \{f(\bar{x})\}.$$

- **Set optimization:** Minimizing a **set-valued** objective function

$$F\colon \Omega \rightrightarrows \mathbb{R}^m, \qquad F(x) \subseteq \mathbb{R}^m .$$

Why Set Optimization?

- transport robots, finance, socio economics, ...
- Bilevel optimization: upper level function
  $F(x) = \{f_u(x, y) \in \mathbb{R}^m \mid y \text{ solves lower level problem } P(x)\}$
- Uncertain values $F(x) = \{f(x)\} + B(0, r(x))$
- Robust multiobjective optimization $F(x) = \{f(x, \xi) \in \mathbb{R}^m \mid \xi \in U\}$, e.g.,
  $F(x) = \{f(x + z) \in \mathbb{R}^m \mid z \in Z\}$ (see later in this talk)

Khan, Tammer, Zălinescu,
Set-valued optimization – an introduction with applications, Springer 2015.

Hamel, Heyde, Löhne, Rudloff, Schrage,
Set Optimization: A Rather Short Introduction, In: Set Optimization and Applications — The State of the Art, Springer 2015.

Outline

- Multiobjective Optimization and Optimality Notions
- Set Optimization and Optimality Notions
- Example: Uncertain Multiobjective Optimization
- Multiobjective Replacements
- Vectorization I (for convex-valued problems)
- Vectorization II
- Uncertain Multiobjective Optimization

Multiobjective Optimization Problem (MOP)

$$\min \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \text{ s.t. } x \in \Omega \qquad \text{(MOP)}$$

with functions $f_j \colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$ and feasible set $\Omega \subseteq \mathbb{R}^n$.

Applications are for instance

- optimal portfolio with minimal risk and maximal return
- optimal design with minimal weight, maximal stability
- optimal treatment plan in medicine which destroys tumour, spares healthy organs
- optimal mixing with minimal energy and maximal mixing quality

Optimal Solutions of a MOP

In general, there is no point $\bar{x} \in \Omega$ with

$$\bar{x} \in \operatorname{argmin}\{f_j(x) \mid x \in \Omega\} \quad \text{for all} \quad j \in \{1, \ldots, m\}$$

at the same time!

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Equivalent conditions in **single-objective** optimization for a minimal solution $\bar{x} \in \Omega$:

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- there is no $x \in \Omega$ with $f(x) < f(\bar{x})$

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- $x \in \Omega$ with $f(x) \leq f(\bar{x})$ implies $f(x) = f(\bar{x})$ (for MOP *efficient point*, next slide)
- there is no $x \in \Omega$ with $f(x) < f(\bar{x})$ (for MOP *weakly efficient point*, soon)

Efficient Points of a MOP

A point $\bar{x} \in X$ is **efficient** for $\min_{x \in \Omega} f(x)$ if it holds for all $x \in \Omega$ with

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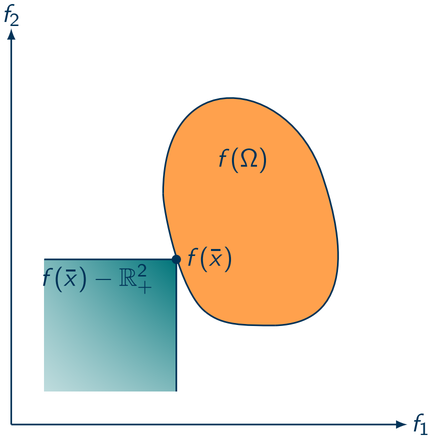
that $f(x) = f(\bar{x})$, i.e., there is no $x \in \Omega$ with $f_i(x) \leq f_i(\bar{x})$, $i = 1, \ldots, m$ and with

$$f_j(x) < f_j(\bar{x}) \quad \text{for at least one } j \in \{1, \ldots, m\}.$$

Equivalently: $(\{f(\bar{x})\} - \mathbb{R}^m_+) \cap f(\Omega) = \{f(\bar{x})\}$.

Then we call $f(\bar{x})$ **nondominated**.



Efficient and Weakly Efficient of a MOP

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A point $\bar{x} \in \Omega$ is **weakly efficient** for $\min_{x \in \Omega} f(x)$ if there is **no** $x \in \Omega$ with

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$$(\{f(\bar{x})\} - \operatorname{int}(\mathbb{R}^m_+)) \cap f(\Omega) = \emptyset \ .$$

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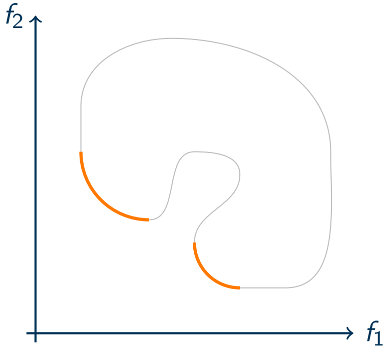
Then we call $f(\bar{x})$ **weakly nondominated** and write $\bar{x} \in \operatorname{argwMin}(f, \Omega, \mathbb{R}^m_+)$.



images of efficient points
nondominated points

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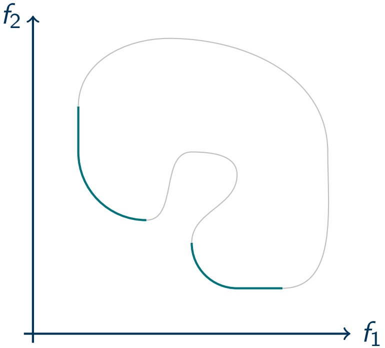

images of weakly efficient points
weakly nondominated points

Approximate Weakly Efficient Points of a MOP

Let $\varepsilon > 0$. A point $\bar{x} \in X$ is ε-**weakly efficient** for $\min_{x \in \Omega} f(x)$ if there is **no** $x \in \Omega$ with

$$f_i(x) \quad < \quad f_i(\bar{x}) - \varepsilon \quad \text{for all } i = 1, \ldots, m,$$

i.e., if

$$(\{f(\bar{x}) - \varepsilon\, e\} - \operatorname{int}(\mathbb{R}^m_+)) \cap f(\Omega) = \emptyset\ .$$

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For $\varepsilon \geq 0$, we write $\varepsilon\text{argwMin}(f, \Omega, \mathbb{R}^m_+)$, and call its elements ε-weakly efficient solutions.
For $\varepsilon = 0$, we write $\text{argwMin}(f, \Omega, \mathbb{R}^m_+)$, and call its elements weakly efficient solutions.

Set Optimization Problem

$$\min_{x \in \Omega} F(x) \tag{SOP}$$

with

- $\Omega \subseteq \mathbb{R}^n$ nonempty and closed,
- $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a given set-valued map such that $\Omega \subseteq \operatorname{dom} F$ and
- $F(x)$ compact for all $x \in \Omega$ (and sometimes: convex)
- sometimes $\bigcup_{x \in \Omega} F(x)$ bounded,
- $C \subseteq \mathbb{R}^m$ a closed, pointed, solid and convex cone, here: $C = \mathbb{R}^m_+$, and $e \in \operatorname{int} C$, here: $e = (1, \ldots, 1)^\top \in \mathbb{R}^m_+$, a given element.

Binary Relations for Set Optimization

We take in the talk as ordering cone $C = \mathbb{R}^m_+$ in $Y = \mathbb{R}^m$, but results apply for any closed, pointed, convex and solid cone $C \subseteq \mathbb{R}^m$.

(i) the u-**less order relation** is defined by: $A \preccurlyeq_u B :\Leftrightarrow A \subseteq B - C$,

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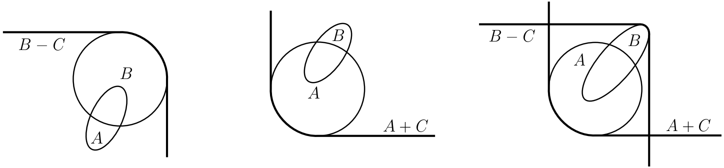
(ii) the *l*-**less order relation** is defined by: $A \preccurlyeq_l B :\Leftrightarrow B \subseteq A + C$,and

(iii) the **set less order relation** is defined by: $A \preccurlyeq_s B :\Leftrightarrow A \preccurlyeq_u B$ and $A \preccurlyeq_l B$.



Optimality Notion in Set Optimization

Definition

Let $* \in \{l, u, s\}$. We denote $\bar{x} \in \Omega$ a **minimal solution** of (SOP*) if

$$\forall\, x \in \Omega : \;\; F(x) \preccurlyeq_* F(\bar{x}) \qquad \Longrightarrow \qquad F(\bar{x}) \preccurlyeq_* F(x).$$

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We denote $\bar{x} \in \Omega$ a **weakly minimal solution** of (SOP*) if there is no $x \in \Omega$ with

$$F(x) \prec_* F(\bar{x})$$

where

$$A \prec_l B :\Longleftrightarrow B \subseteq A + \operatorname{int}(\mathbb{R}^m_+), \quad A \prec_u B :\Longleftrightarrow A \subseteq B - \operatorname{int}(\mathbb{R}^m_+)$$

$$A \prec_s B :\Longleftrightarrow A \prec_l B \wedge A \prec_u B.$$

Uncertain Multiobjective Optimization

Let:

- $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ be continuous,
- $\Omega \subseteq \mathbb{R}^n$ be nonempty and closed,
- $\mathcal{U} \subseteq \mathbb{R}^k$ be nonempty and compact (the so called uncertainty set).

The uncertain multiobjective problem associated to this data is:

$$\left\{ \begin{array}{ll} \min\limits_{x} & f(x,u) \\ \text{s.t.} & x \in \Omega \end{array} \;\middle|\; u \in \mathcal{U} \right\} \tag{$\mathcal{UMP}$}$$

The Scalar Case

Let $m = 1$.

$$\left\{ \begin{array}{ll} \min\limits_{x} & f(x,u) \\ \text{s.t.} & x \in \Omega \end{array} \;\middle|\; u \in \mathcal{U} \right\} \qquad (\mathcal{UMP})$$

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Robust counterpart problem:

$$\begin{array}{ll} \min\limits_{x} & \sup\limits_{u \in \mathcal{U}} f(x,u) \\ \text{s.t.} & x \in \Omega \end{array} \qquad (\mathcal{RCP})$$

Solutions of $(\mathcal{RCP})$ are called robust for $(\mathcal{UMP})$.

Robust Counterpart Problem for $m \geq 2$

Consider $F_{\mathcal{U}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by $F_{\mathcal{U}}(x) := \{f(x,u) \in \mathbb{R}^m \mid u \in \mathcal{U}\}$.

Robust counterpart problem:

$$\min_x \; F_{\mathcal{U}}(x) \quad \text{s.t.} \; x \in \Omega \tag{$\mathcal{RCP}$}$$

Recall: $\bar{x} \in \Omega$ is a weakly minimal solution of the set optimization problem $(\mathcal{RCP})$ w.r.t. $* = u$ if

$$\nexists \, x \in \Omega : F_{\mathcal{U}}(x) \subseteq F_{\mathcal{U}}(\bar{x}) - \operatorname{int}(\mathbb{R}^m_+).$$

Uncertain Multiobjective Optimization Problem

$$\left\{ \begin{array}{ll} \min\limits_{x} & f(x,u) \\ \text{s.t.} & x \in \Omega \end{array} \;\middle|\; u \in \mathcal{U} \right\} \qquad (\mathcal{UMP})$$

Definition (Ehrgott, Ide, Schöbel 2014, Ide, Köbis, Kuroiwa, Schöbel, Tammer 2014)

$\bar{x} \in \Omega$ is a **robust weakly minimal solution** of $(\mathcal{UMP})$ if it is a weakly minimal solution of $(\mathcal{RCP})$ (w.r.t. $* = u$), i.e,

$$\nexists\, x \in \Omega : F_{\mathcal{U}}(x) \subseteq F_{\mathcal{U}}(\bar{x}) - \operatorname{int}(\mathbb{R}^m_+),$$

where

$$F_{\mathcal{U}}(x) = \{f(x,u) \in \mathbb{R}^m \mid u \in \mathcal{U}\}.$$

State of the Art for Set Optimization

There are just a few approaches to numerically solve set optimization problems, for instance

- for polyhedral convex sets [Schrage, Löhne 2013]
- scalarization based, e.g., [Köbis, Köbis 2016]
- for finite families of sets [Günther, Köbis, Popovici 2019]
- derivative-free descent method [Jahn, 2015]

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We propose approaches based on solving (finite dimensional) multiobjective replacement problems.

Replacement Problem for Ball-valued maps

Theorem

Let $c\colon \Omega \to \mathbb{R}^m$ and $r\colon \Omega \to \mathbb{R}_+$. Let the ball-valued map $F\colon \Omega \rightrightarrows \mathbb{R}^m$ be defined by

$$F(x) := \{c(x)\} + \{y \in \mathbb{R}^m \mid \|y\|_2 \leq r(x)\} \text{ for all } x \in \Omega.$$

Then $\bar{x} \in \Omega$ is a minimal solution of (SOP^s) if and only if $\bar{x}$ is an efficient solution of

$$\min_{x \in \Omega} \begin{pmatrix} I_m & e \\ I_m & -e \end{pmatrix} \begin{pmatrix} c(x) \\ r(x) \end{pmatrix}$$

w.r.t. the ordering cone $\mathbb{R}^{2m}_+$, where I_m is the m-dimensional identity matrix and e is the m-dimensional all-one vector.

How to Solve Set Optimization Problems?

For $F(x)$ convex for all $x \in \Omega$, similar to a result in [Jahn, 2015], it holds for $x^1, x^2 \in \Omega$:

$$F(x^1) \preccurlyeq_l F(x^2) \;\Leftrightarrow\; \forall\, \ell \in \mathbb{R}^m_+ \setminus \{0\} : \quad \min_{y \in F(x^1)} \ell^\top y \le \min_{y \in F(x^2)} \ell^\top y \,.$$

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Theorem

(a) *$\bar{x} \in \Omega$ is a minimal solution of (SOP^l) if and only if there is no $x \in \Omega$ such that*

$$\forall \ell \in \mathbb{R}^m_+ \setminus \{0\} : \min_{y \in F(x)} \ell^\top y \leq \min_{\bar{y} \in F(\bar{x})} \ell^\top \bar{y} \;\; \textit{and} \;\; \exists \hat{\ell} \in \mathbb{R}^m_+ \setminus \{0\} : \min_{y \in F(x)} \hat{\ell}^\top y < \min_{\bar{y} \in F(\bar{x})} \hat{\ell}^\top \bar{y}.$$

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(b) *$\bar{x} \in \Omega$ is a weakly minimal solution of (SOP^l) if and only if there is no $x \in \Omega$ such that*

$$\forall \ell \in \mathbb{R}^m_+ \setminus \{0\} : \min_{y \in F(x)} \ell^\top y < \min_{\bar{y} \in F(\bar{x})} \ell^\top \bar{y}.$$

Minimal Value Function

Let $\ell \in \mathbb{R}^m_+ \setminus \{0\}$ be given. The corresponding **minimal value function** $\ell_{\min} : \Omega \to \mathbb{R}$ is defined by

$$\ell_{\min}(x) := \min_{y \in F(x)} \ell^\top y.$$

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A simple first sufficient condition for a minimal solution $\bar{x} \in \Omega$:
If it holds $\ell_{\min}(\bar{x}) < \ell_{\min}(x)$ for all $x \in \Omega \setminus \{\bar{x}\}$, then $\bar{x}$ is a minimal solution of (SOP^l).

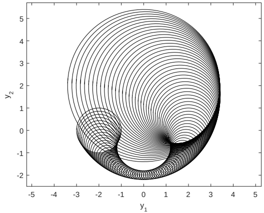
Hence, by solving $\min_{x \in \Omega} \ell_{\min}(x)$ we can determine (weakly) minimal solutions of the set optimization problem!

Example I

$F : [\pi, \frac{5}{2}\pi] \rightrightarrows \mathbb{R}^2$ with

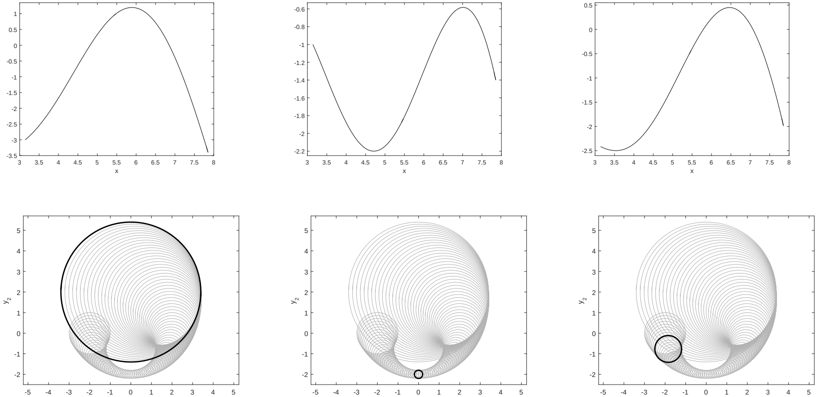
$$F(x) = \left\{ y \in \mathbb{R}^2 \;\middle|\; y = 2\begin{pmatrix}\cos(x)\\ \sin(x)\end{pmatrix} + r\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix},\; r \in [0, R(x)],\; t \in [0, 2\pi] \right\}$$

where the radii of the balls are given by $R(x) = 1 + \frac{4}{5}((\frac{2}{\pi}x - 3)^2 - 1)$.



Example II

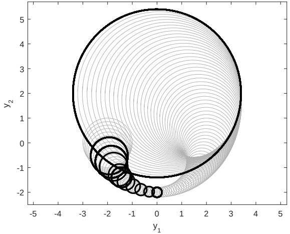
For $\ell^1 := (1,0)^\top$, $\ell^2 := (0,1)^\top$, $\ell^3 := \frac{1}{\sqrt{2}}(1,1)^\top$: the graphs of $\ell^i_{\min}$ and the sets $F(\bar{x}^i)$ to the minimal solutions $\bar{x}^i$.



Main Idea:

Study

$$\min_{x\in\Omega}\begin{pmatrix}\ell^1_{\min}(x)\\ \ell^2_{\min}(x)\\ \ell^3_{\min}(x)\end{pmatrix}=\begin{pmatrix}\min_{y\in F(x)}(\ell^1)^\top y\\ \min_{y\in F(x)}(\ell^2)^\top y\\ \min_{y\in F(x)}(\ell^3)^\top y\end{pmatrix}$$



The multiobjective replacement problem

To a finite nonempty list $\mathcal{L} = \{\ell^1, \dots, \ell^k\} \subseteq \{y \in \mathbb{R}^m_+ \mid \|y\| = 1\}$ we assign the **multiobjective optimization problem**

$$\min_{x \in \Omega} f_{\mathcal{L}}(x) \tag{MOP$_{\mathcal{L}}$}$$

with $f^{\mathcal{L}} := (\ell^1_{\min}, \dots, \ell^k_{\min})^\top : \mathbb{R}^n \to \mathbb{R}^k$, and, as before, $\ell^i_{\min} : \Omega \to \mathbb{R}$,

$$\ell^i_{\min}(x) := \min_{y \in F(x)} (\ell^i)^\top y$$

for all $i \in \{1, \dots, k\}$.

- Gerlach, Rocktäschel, *On convexity and quasiconvexity of extremal value functions in set optimization*, Applied Set-Valued Analysis and Optim., 2021.
- Eichfelder, Gerlach, Rocktäschel, *Convexity and continuity of specific set-valued maps and their extremal value functions*, J. of Applied and Numerical Optim., 2022.

How to Find Weakly Minimal Solutions of (SOP)?

Let $\mathcal{L} = \{\ell^1, \ldots, \ell^k\} \subseteq \{y \in \mathbb{R}^m_+ \mid \|y\| = 1\}$.

Theorem

The weakly efficient solutions of $(\mathrm{MOP}_\mathcal{L})$ *are weakly minimal solutions of (SOP^l), i.e.,*

$$\mathrm{argwMin}(f_\mathcal{L}, \Omega, \mathbb{R}^k_+) \subseteq \mathrm{argwMin}^l(F, \Omega, \mathbb{R}^m_+).$$

We do not have:

- $\mathrm{argMin}(f_\mathcal{L}, \Omega, \mathbb{R}^k_+) \subseteq \mathrm{argMin}^l(F, \Omega, \mathbb{R}^m_+)$
- $\mathrm{argwMin}(f_\mathcal{L}, \Omega, \mathbb{R}^k_+) \supseteq \mathrm{argwMin}^l(F, \Omega, \mathbb{R}^m_+)$

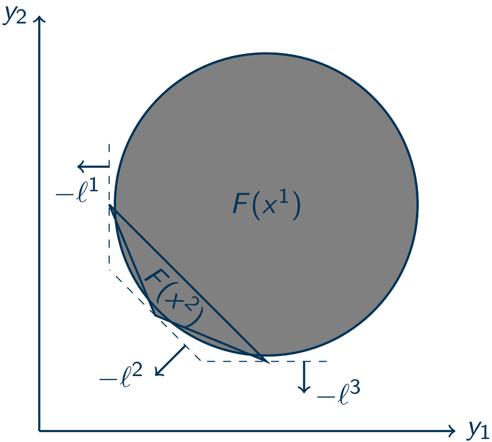
Weakly Minimal Solutions of (SOP) - Example

Example

- Choose $\mathcal{L} = \{(1,0)^\top, \ (\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}})^\top, (0,1)^\top\}$ (three or more)
- $\Omega = \{x^1, x^2\}$
- $F(x^1) = \{y \in \mathbb{R}^2 \mid ||y||_2 \leq 1\}$
- $F(x^2) = \text{conv}(\{(1+\varepsilon)(-1,0)^\top, (1+\varepsilon)(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})^\top, (1+\varepsilon)(0,-1)^\top\})$, where

$$\varepsilon := \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{4}\min_{i \neq j} ||\ell^i - \ell^j||_2^2}\right)$$

Weakly Minimal Solutions of (SOP) - Example



Is There a Nice 'ε-connection'?

Theorem

For every $\varepsilon > 0$ *there exists a finite* $\mathcal{L} = \mathcal{L}(\varepsilon)$ *such that,*

$$\mathrm{argwMin}^{l}(F, \Omega, \mathbb{R}^m_+) \subseteq \varepsilon\mathrm{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^{|\mathcal{L}|}_+).$$

Is There a Nice 'ε-connection'?

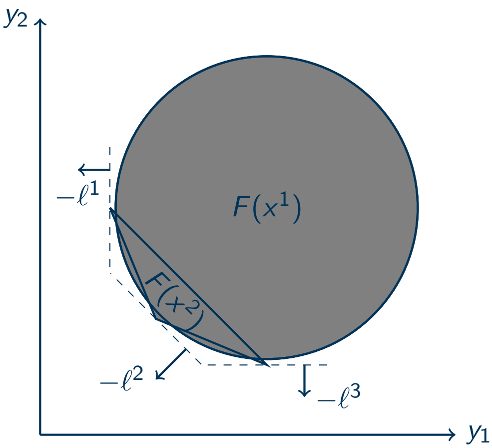
Theorem

For every $\varepsilon > 0$ there exists a finite $\mathcal{L} = \mathcal{L}(\varepsilon)$ such that,

$$\text{argwMin}^I(F, \Omega, \mathbb{R}^m_+) \subseteq \varepsilon\text{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^{|\mathcal{L}|}_+).$$

It follows:

$$\text{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^{|\mathcal{L}|}_+) \subseteq \text{argwMin}^I(F, \Omega, \mathbb{R}^m_+) \subseteq \varepsilon\text{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^{|\mathcal{L}|}_+).$$



- $x^1 \in \text{argwMin}(F, \Omega, \mathbb{R}^2_+)$
- $x^1 \notin \text{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^3_+)$
- $x^2 \in \text{argwMin}(F, \Omega, \mathbb{R}^2_+)$
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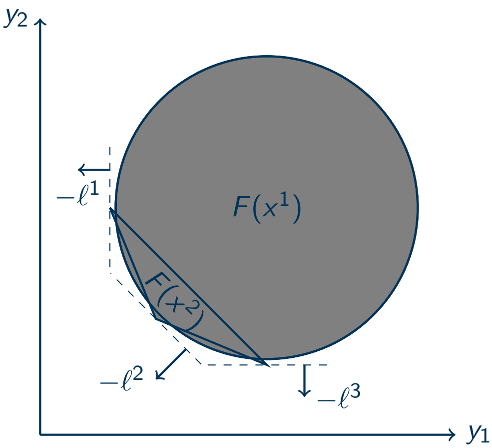

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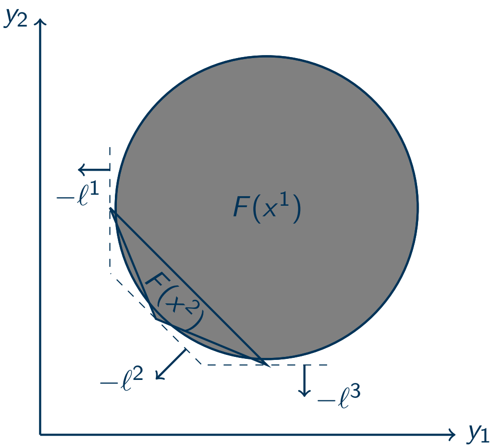
- We see: $x^1 \in \varepsilon\operatorname{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^3_+)$, $\varepsilon \approx 0.04$



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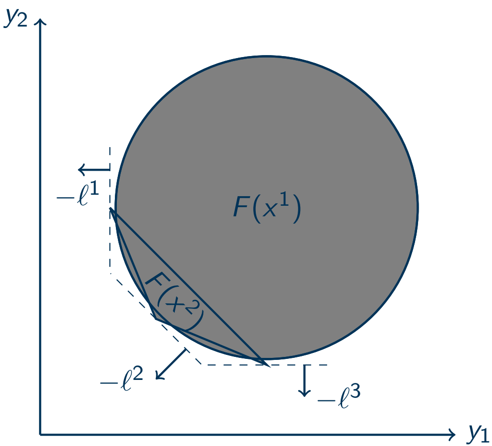

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for $\varepsilon \approx 1.12$

A Hint on Choosing $\mathcal{L}$

Theorem

Suppose that

$$\bar{u} := \sup \left\{ \|y\| \mid y \in \bigcup_{x \in \Omega} \operatorname{Min}(F(x), \mathbb{R}^m_+) \right\} < +\infty.$$

Let $\varepsilon > 0$ be given and $\mathcal{L}$ be a finite set with

$$\{y \in \mathbb{R}^m_+ \mid \|y\| = 1\} \subset \mathcal{L} + \frac{\varepsilon}{4\bar{u}} \mathbb{B}.$$

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Then

$$\mathrm{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^k_+) \subseteq \mathrm{argwMin}^l(F, \Omega, \mathbb{R}^m_+) \subseteq \varepsilon\mathrm{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^k_+).$$

Finite Dimensional Vectorization Property (FDVP)

Definition

We say that $(\mathrm{MOP}_{\mathcal{L}})$ satisfies the **finite dimensional vectorization property (FDVP)** if

$$\forall x \in \operatorname{argwMin}^{l}(F, \Omega, \mathbb{R}^m_+) \; \exists \mathcal{L} \subseteq \mathbb{R}^m_+ \setminus \{0\} \colon |\mathcal{L}| < \infty \wedge x \in \operatorname{argwMin}(f_{\mathcal{L}}, \Omega, \mathbb{R}^{|\mathcal{L}|}_+)$$

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Theorem

Let Ω be convex, $\Omega \subseteq \mathrm{int}(\mathrm{dom} F)$. If $\mathrm{gph} F$ is convex, then $(\mathrm{MOP}_{\mathcal{L}})$ satisfies (FDVP).

And Other Set Order Relations?

For $F(x)$ convex for all $x \in \Omega$, similar to a result in [Jahn, 2015], it holds for $x^1, x^2 \in \Omega$:

$$F(x^1) \preccurlyeq_u F(x^2) \;\Leftrightarrow\; \forall\, \ell \in \mathbb{R}^m_+ \setminus \{0\} : \quad \max_{y \in F(x^1)} \ell^\top y \leq \max_{y \in F(x^2)} \ell^\top y\ .$$

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Theorem

(a) *$\bar{x} \in \Omega$ is a minimal solution of (SOP^u) if and only if there is no $x \in \Omega$ such that*

$$\forall \ell \in \mathbb{R}^m_+ \setminus \{0\} : \max_{y \in F(x)} \ell^\top y \leq \max_{\bar{y} \in F(\bar{x})} \ell^\top \bar{y} \;\; \textit{and} \;\; \exists \hat{\ell} \in \mathbb{R}^m_+ \setminus \{0\} : \max_{y \in F(x)} \hat{\ell}^\top y < \max_{\bar{y} \in F(\bar{x})} \hat{\ell}^\top \bar{y}.$$

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(MOP) for Other Set Relations

For finite, nonempty sets $\mathcal{L} = \{\ell^1, \dots, \ell^p\}, \mathcal{U} = \{\ell^{p+1}, \dots, \ell^{p+q}\} \subseteq \mathbb{R}^m_+ \setminus \{0\}$ we define the multiobjective optimization problem:

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$$\min_{x \in \Omega} f_{\mathcal{L},\mathcal{U}}(x) := \begin{pmatrix} \min_{y \in F(x)} \ell^1(y) \\ \vdots \\ \min_{y \in F(x)} \ell^p(y) \\ \max_{y \in F(x)} \ell^{p+1}(y) \\ \vdots \\ \max_{y \in F(x)} \ell^{p+q}(y) \end{pmatrix} \text{ w.r.t. } \mathbb{R}^{p+q}_+ \qquad (\mathrm{MOP}(\mathcal{L},\mathcal{U}))$$

And for Nonconvex Sets $F(x)$? Motivation:

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Definition

$\bar{x}$ is a **vector approach weakly minimal solution** of (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a weakly efficient solution of the **multiobjective optimization problem**

$$\begin{aligned} \min_{x,y}\ & y \\ \text{s.t.}\ & (x, y) \in \operatorname{gph} F, \\ & x \in \Omega. \end{aligned} \qquad (\mathcal{MP}_1)$$

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We know that:

$$\bar{x} \text{ vector approach weakly minimal solution} \Longrightarrow \bar{x} \in \operatorname{argwMin}^l(F, \Omega, \mathbb{R}^m_+).$$

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$$\operatorname{argwMin}_x (\mathcal{MP}_1) := \{x \in \mathbb{R}^n \mid \exists\, y \in \mathbb{R}^m : (x, y) \in \operatorname{argwMin} (\mathcal{MP}_1)\} \subseteq \operatorname{argwMin}^l(F, \Omega, \mathbb{R}^m_+).$$

Vectorization II—Motivation

$$\min_{x,y^1,y^2} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$$

$$\text{s.t } (x, y^1) \in \text{gph } F, \quad (x, y^2) \in \text{gph } F, \quad x \in \Omega. \qquad (\mathcal{MP}_2)$$

We know that:

$$\text{argwMin}_x\,(\mathcal{MP}_1) \subseteq \text{argwMin}^l(F, \Omega, \mathbb{R}^m_+).$$

Vectorization Scheme

For $p \in \mathbb{N}$:

$$\min_{x,y^1,\dots,y^p} \begin{pmatrix} y^1 \\ \vdots \\ y^p \end{pmatrix}$$

$$\begin{aligned} \text{s.t } & (x, y^1) \in \operatorname{gph} F, \\ & \vdots \\ & (x, y^p) \in \operatorname{gph} F, \\ & x \in \Omega. \end{aligned} \qquad (\mathcal{MP}_p)$$

Question:

$$\operatorname{argwMin}_x (\mathcal{MP}_p) \overset{?}{\subseteq} \operatorname{argwMin}^l(F, \Omega, \mathbb{R}^m_+).$$

Relationships between $(\mathcal{MP}_p)$ and (SOP^l)

Theorem

The following inclusions hold:

$$\bigcup_{p\in\mathbb{N}} \operatorname{argwMin}_x(\mathcal{MP}_p) \subseteq \operatorname{argwMin}^l(F,\Omega,\mathbb{R}^m_+) = \bigcap_{\varepsilon>0}\bigcup_{p\in\mathbb{N}} \varepsilon\operatorname{argwMin}_x(\mathcal{MP}_p).$$

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Corollary

$$\forall\, \varepsilon > 0, \exists\, p \in \mathbb{N} : \ \operatorname{argwMin}_x (\mathcal{MP}_p) \subseteq \operatorname{argwMin}^l(F,\Omega,\mathbb{R}^m_+) \subseteq \varepsilon \operatorname{argwMin}_x (\mathcal{MP}_p).$$

Solutions of $(\mathcal{MP}_p)$ in the Image Space

We have:

$$\bigcup_{p\in\mathbb{N}} \operatorname{argwMin}_x (\mathcal{MP}_p) \subseteq \operatorname{argwMin}^l(F, \Omega, \mathbb{R}^m_+)$$

Theorem

Suppose that Ω is compact and gph F *is closed. Then,*

$$\forall\, x \in \Omega,\ \exists\, \bar{x} \in \operatorname{cl}\left(\bigcup_{p\in\mathbb{N}} \operatorname{argwMin}_x (\mathcal{MP}_p)\right) :\ F(\bar{x}) \preceq^l F(x).$$

Finite Dimensional Vectorization Property

We have:

$$\bigcup_{p \in \mathbb{N}} \operatorname{argwMin}_x \left(\mathcal{MP}_p\right) \subseteq \operatorname{argwMin}^l(F, \Omega, \mathbb{R}^m_+)$$

Finite Dimensional Vectorization Property

We have:

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Definition

We say that (SOP^l) satisfies the **finite dimensional vectorization property (FDVP)** if

$$\exists\, p\in\mathbb{N} : \operatorname{argwMin}_x\left(\mathcal{MP}_p\right) = \operatorname{argwMin}^l(F,\Omega,\mathbb{R}^m_+).$$

(FDVP) for (SOP^l): Discrete Case

Theorem

(a) *Suppose that* $|\Omega| < +\infty$. *Then,*

$$(SOP^l) \text{ satisfies } (FDVP) \text{ with } p = |\Omega| - 1.$$

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Example: $F(x) := \{f(x,u) \mid u \in \mathcal{U}\}$, where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^m$ and $|\mathcal{U}| < \infty$.

(FDVP) for (SOP^l): Polytope Case

Theorem

Suppose that F is polytope-valued and that $\sup_{x\in\Omega} |\text{ext}(F(x))| < +\infty$. Then,

$$(SOP^l) \text{ satisfies (FDVP) with } p = \sup_{x\in\Omega} |\text{ext}(F(x))|.$$

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Example: $F(x) := \{y \in \mathbb{R}^m \mid Ay \leq f(x)\}$, where $A \in \mathbb{R}^{k\times m}$ and $f : \mathbb{R}^n \to \mathbb{R}^k$.

(FDVP) for (SOP^l): Convex Case

Theorem

Suppose additionally that Ω is convex, $\operatorname{gph} F$ is convex, F is locally bounded around any point in Ω.
Then,

$$(SOP^l) \text{ satisfies (FDVP) with } p := n + 1.$$

Scalarization of $(\mathrm{MOP}_{\mathcal{L}})$-Relation to $(\mathcal{MP}_p)$

For $v \in \mathbb{R}^p_+$ consider

$$\min_{x \in \Omega} \left(v_1 \min_{y \in F(x)} \ell^1(y) + \ldots + v_p \min_{y \in F(x)} \ell^p(y) \right)$$

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$$= \min_{x\in\Omega}\left(\min_{y^1\in F(x)} (v_1\ell^1)(y^1) + \ldots + \min_{y^p\in F(x)} (v_p\ell^p)(y^p)\right)$$

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For $v \in \mathbb{R}^p_+$ consider

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&\min_{x\in\Omega}\left(v_1 \min_{y\in F(x)} \ell^1(y) + \ldots + v_p \min_{y\in F(x)} \ell^p(y)\right)\\
&= \min_{x\in\Omega}\left(\min_{y^1\in F(x)} (v_1\ell^1)(y^1) + \ldots + \min_{y^p\in F(x)} (v_p\ell^p)(y^p)\right)\\
&= \min_{(x,y^1,\ldots,y^p)\in \operatorname{gph} F^p} w^\top (y^1,\ldots,y^p)^\top,
\end{aligned}
$$

where $w := (v_1\ell^1,\ldots,v_p\ell^p)^\top \in (\mathbb{R}^m_+)^p$ and
$\operatorname{gph} F^p := \{(x,y^1,\ldots,y^p) \mid \forall i \in [p] : (x,y^i) \in \operatorname{gph} F\}$.

Uncertain Multiobjective Optimization Problem

Definition (Ehrgott et al. 2014)

$\bar{x} \in \Omega$ is a robust weakly minimal solution of $(\mathcal{UMP})$ if it is a solution of $(\mathcal{RCP})$, i.e.,

$$\nexists\, x \in \Omega : F_{\mathcal{U}}(x) \prec_u F_{\mathcal{U}}(\bar{x}),$$

where $F_{\mathcal{U}}(x) = \{f(x,u) \mid u \in \mathcal{U}\}$. The set of robust weakly minimal solutions is denoted by argwMin$(\mathcal{UMP})$.

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$$A \prec_u B \Longleftrightarrow \left(A - \mathbb{R}^m_+\right)^c \prec_l \left(B - \mathbb{R}^m_+\right)^c .$$

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Definition (Ehrgott et al. 2014)

$\bar{x} \in \Omega$ is a robust weakly minimal solution of $(\mathcal{UMP})$ if it is a solution of $(\mathcal{RCP})$, i.e.,

$$\nexists\, x \in \Omega : F_{\mathcal{U}}(x) \prec_u F_{\mathcal{U}}(\bar{x}),$$

where $F_{\mathcal{U}}(x) = \{f(x,u) \mid u \in \mathcal{U}\}$. The set of robust weakly minimal solutions is denoted by $\operatorname{argwMin}(\mathcal{UMP})$.

$$A \prec_u B \Longleftrightarrow \left(A - \mathbb{R}^m_+\right)^c \prec_l \left(B - \mathbb{R}^m_+\right)^c .$$

We need something like ...see Part 2

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