# Stochastic Models and Optimal Control of Epidemics under Partial Information

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Joint work with

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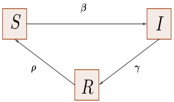
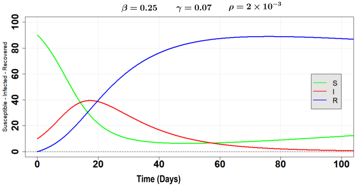
# Introduction

## SIRS Model (Kermack and McKendrick (1927))

$$\begin{aligned}
\dot{S}(t) &= -\beta S(t)\frac{I(t)}{N} + \rho R(t) && \text{Susceptible}\\
\dot{I}(t) &= \beta S(t)\frac{I(t)}{N} - \gamma I(t) && \text{Infected}\\
\dot{R}(t) &= \gamma I(t) - \rho R(t) && \text{Recovered}
\end{aligned}$$




- System of ODEs for $X = (S, I, R)^\top : \dot{X}(t) = F(X(t))$
- Basic reproduction number $\mathcal{R}_0 = \dfrac{\beta}{\gamma}$

  Effective reproduction number $\mathcal{R}(t) = \dfrac{\beta}{\gamma}\dfrac{S(t)}{N}$

# ODE Models: Properties and Limitations

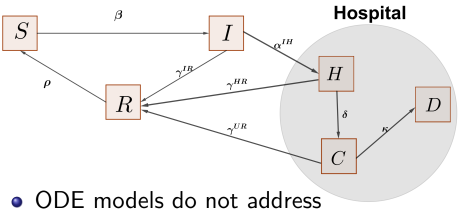
- Description and prediction of relative subpopulation sizes and
  "average" absolute subpopulation sizes

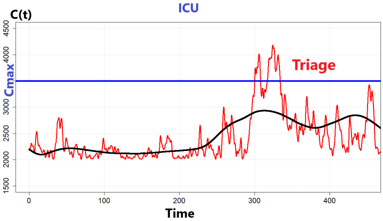
  for large total populations size $N$
- No information about deviations from the average.

  Interesting for absolute subpopulation sizes and models with

  small and moderate total population size $N$,

  small compartments such as hospitals, intensive care units (ICU).




- ODE models do not address

  uncertain parameters and initial conditions,

  forecast uncertainties,

  not directly observable subpopulation sizes

  (partial information, statistical learning of dark figures, nowcast uncertainties).

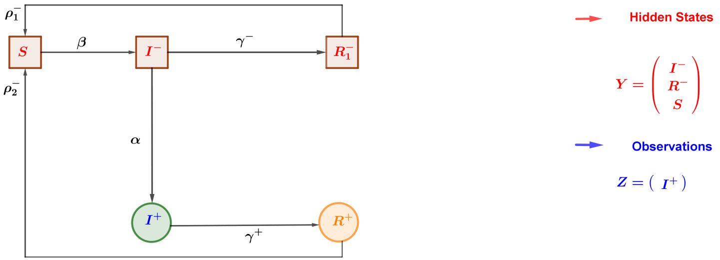
# COVID-19 Model with Partial Information

$I^-$ Infected, non-detected

$R^-$ Recovered, non-detected

S Susceptible

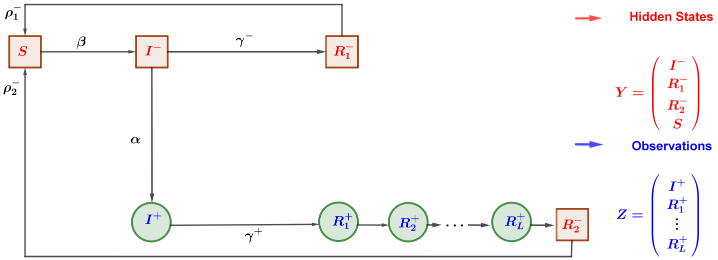
$I^+$ Infected, detected




Charpentier et al. (2020), Meyer-Hermann et al. (2021)

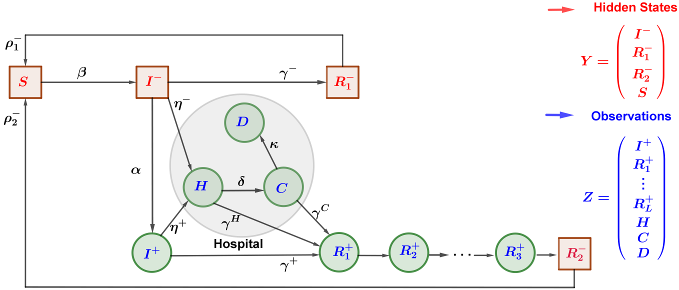
# COVID-19 Model with Partial Information

| | | | |
|---|---|---|---|
| $I^-$ | Infected, non-detected | $I^+$ | Infected, detected |
| $R_1^- / R_2^-$ | Recovered, non-detected/fading immunity | $R_1^+, \ldots, R_L^+$ | Recovered, detected |
| S | Susceptible | | |



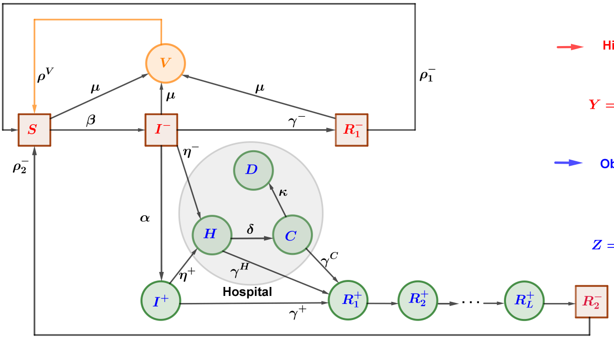
# COVID-19 Model with Hospital

| | | | |
|---|---|---|---|
| $I^-$ | Infected, non-detected | $I^+$ | Infected, detected |
| $R_1^- / R_2^-$ | Recovered, non-detected/fading immunity | $R_1^+, \ldots, R_L^+$ | Recovered, detected |
| S | Susceptible | $H$ | Hospitalized |
| | | $C$ | Intensive Care Unit |
| | | $D$ | Death |



# COVID-19 Model with Hospital and Vaccination

| | | | |
|---|---|---|---|
| $I^-$ | Infected, non-detected | $I^+$ | Infected, detected |
| $R_1^-/R_2^-$ | Recovered, non-detected/fading immunity | $R_1^+, \ldots, R_L^+$ | Recovered, detected |
| S | Susceptible | $H$ | Hospitalized |
| | | $C$ | Intensive Care Unit |
| | | $D$ | Death |

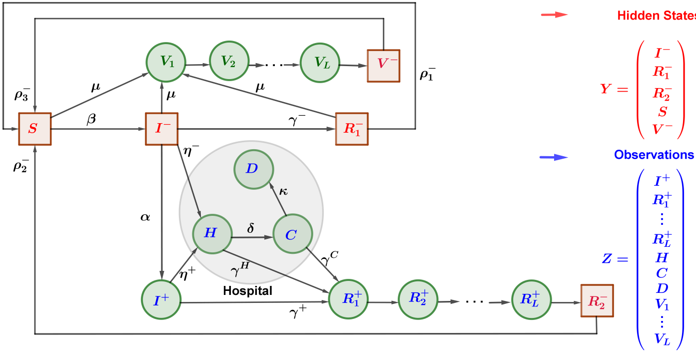



$$Y = \begin{pmatrix} I^- \\ R_1^- \\ R_2^- \\ S \\ V \end{pmatrix}$$

$$Z = \begin{pmatrix} I^+ \\ R_1^+ \\ \vdots \\ R_L^+ \\ H \\ C \\ D \end{pmatrix}$$

# COVID-19 Model with Hospital and Vaccination

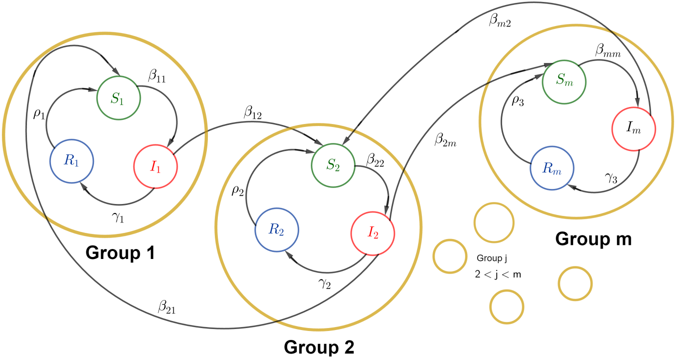
| | | | |
|---|---|---|---|
| $I^-$ | Infected, non-detected | $I^+$ | Infected, detected |
| $R_1^- / R_2^-$ | Recovered, non-detected/fading immunity | $R_1^+, \dots, R_1^+$ | Recovered detected |
| S | Susceptible | $H$ | Hospitalized |
| $V^-$ | Vaccinated, fading immunity | $C$ | Intensive Care Unint |
| | | $D$ | Death |
| | | $V_1, \dots, V_L$ | Vaccinated |




Parameters $\beta$, $\alpha$, $\mu$ may be time-dependent and controlled

# Multi-Group Models

Models with several regions, age groups, vaccination states, ...

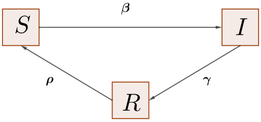


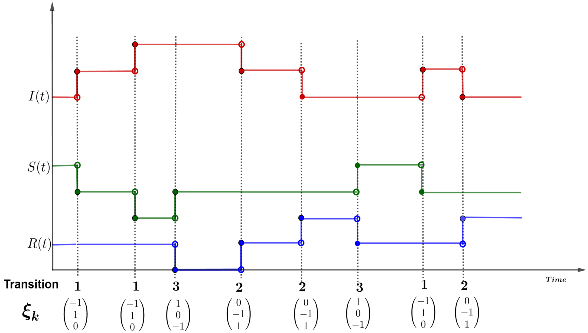
# Microscopic Stochastic Epidemic Models

- Similar to models of chemical reaction networks (Anderson, Kurtz (2011))
- Divide population of size $N$ into $d$ compartments
- $X_i(t) \in \{0, \ldots, N\}$ absolute size of subpopulation in compartment $i = 1, \ldots, d$
- $\overline{X}_i(t) = \frac{1}{N} X_i(t) \in [0, 1]$ relative size of subpopulation
- State $X = X(t) = (X_1, \ldots, X_d)^\top$ (or $\overline{X} = (\overline{X}_1, \ldots, \overline{X}_d)^\top$ )
- Individuals may undergo $K \in \mathbb{N}$ transitions between compartments
- Transition vectors $\xi_k = \Delta X(t) = X(t) - X(t-) \in \mathbb{Z}^d$ if the transition $k$ occurs at time $t$, $k = 1, \ldots, K$, typical entries of $\xi_k$ are $-1, 0, +1$
- Counting processes $M_k(t)$ : number of transition $k$ in $[0, t]$

$$X(t) = X(0) + \sum_{k=1}^{K} \xi_k M_k(t)$$

# Example: Stochastic SIRS Model




$$d = 3, \quad X = (S, I, R)^\top, \quad K = 3 \text{ transitions}$$

| k | Transition | Transition vectors $\xi_k$ | intensity $\lambda_i(x)$ |
|---|---|---|---|
| 1 | Infection of susceptible | $(-1, \quad 1, \quad 0)^\top$ | $\beta S \frac{I}{N} = \beta x_1 \frac{x_2}{N}$ |
| 2 | Recovering of infected | $(\quad 0, -1, \quad 1)^\top$ | $\gamma I = \gamma x_2$ |
| 3 | Losing immunity | $(\quad 1, \quad 0, -1)^\top$ | $\rho R = \rho x_3$ |

# Microscopic Stochastic Epidemic Models (cont.)

Recall: $X^N(t) = X^N(0) + \sum_{k=1}^{K} \xi_k M_k(t)$ state for population of size $N$

Assume $\mathbb{P}(M_k(t+\Delta t) - M_k(t) = 1 | X^N(t)) = \lambda_k(X^N(t))\Delta t + o(\Delta t)$

Describe counting processes $M_k$ by independent Poisson processes

## Continuous–Time Markov Chain (CTMC)

$$X^N(t) = X^N(0) + \sum_{k=1}^{K} \xi_k \Pi_k\Big( \int_0^t \lambda_k(X^N(s))ds \Big)$$

where $\Pi_1, \ldots, \Pi_K$ are independent standard Poisson processes

State-dependent intensities $\lambda_k = \lambda_k(X^N(s))$, for $k = 1, \cdots, K$

Assume scaling property $\lambda_k(x) = \lambda_k^N(x) = N\nu_k(N^{-1}x)$
where $\nu_k(z)$ is the intensity
in terms of relative subpopulation size $z = N^{-1}x$
and independent of $N$

Intensities may depend on time: $\lambda_k = \lambda_k(t, x)$

# Macroscopic Models & Large Population Limit

## Law of Large Numbers for Poisson Process

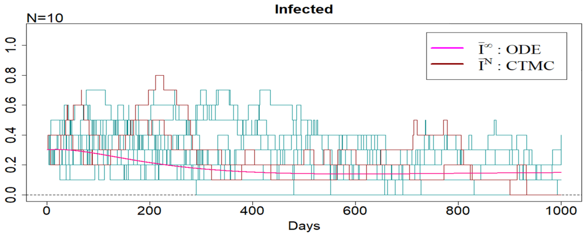
$$\left|\frac{1}{N}\Pi(Nu) - u\right| \xrightarrow[N\to\infty]{} 0 \quad \text{a.s., uniformly} \quad \text{for} \quad \text{all } u \leq u_0$$

implies $\overline{X}^N(t) \xrightarrow[N\to\infty]{} \overline{X}^\infty(t)$ uniformly for all $t \leq T$

$\overline{X}^\infty$ satisfies ODE Anderson & Kurtz (2011), Britton & Pardoux (2018)

## Large Population Limit for Relative Subpopulation Sizes

$$\frac{d}{dt}\overline{X}^\infty(t) = \overline{F}(t, \overline{X}^\infty(t)) \quad \textit{with} \quad \overline{F}(t,z) = \sum_{k=1}^{K} \xi_k \nu_k(t,z)$$



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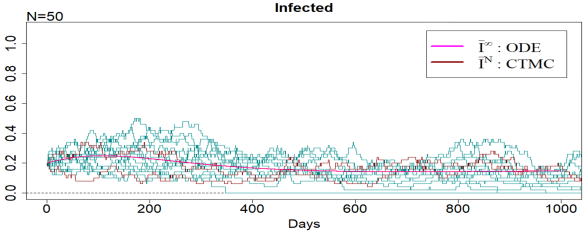

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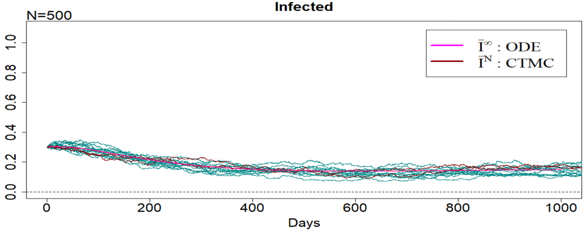

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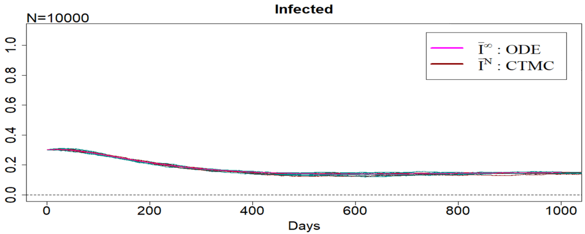

# Macroscopic Models & Diffusion Approximation

## Central Limit Theorem for Poisson Process

$$\frac{1}{\sqrt{N}}(\Pi(Nu) - Nu) \underset{N\to\infty}{\Longrightarrow} W(u) \quad \text{Brownian motion}$$

Scaling property $W(\int_0^t a(s)ds) \overset{d}{=} \int_0^t \sqrt{a(s)}dW(s)$

## Diffusion Approximation of $X^N$ and $\overline{X}^N$ by SDEs

$$\begin{array}{llll} \text{Absolute} & dX^D(t) & = F(t, X^D(t))dt & + \sigma(t, X^D(t))dW(t) \\ \text{Relative} & d\overline{X}^D(t) & = \overline{F}(t, \overline{X}^D(t))dt & + \frac{1}{\sqrt{N}}\overline{\sigma}(t, \overline{X}^D(t))dW(t) \end{array}$$

with $F(t,x) = \sum_{k=1}^{K} \xi_k \lambda_k(t,x)$, $\sigma(t,x) = (\xi_1\sqrt{\lambda_1(t,x)}, \dots, \xi_K\sqrt{\lambda_K(t,x)})$

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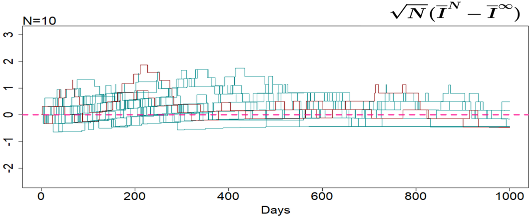

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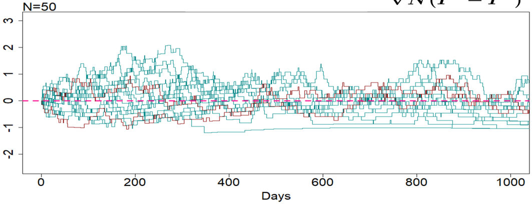
Relative $d\overline{X}^D(t) = \overline{F}(t, \overline{X}^D(t))dt + \frac{1}{\sqrt{N}}\overline{\sigma}(t, \overline{X}^D(t))dW(t)$

with $F(t,x) = \sum_{k=1}^{K} \xi_k \lambda_k(t,x)$, $\sigma(t,x) = (\xi_1\sqrt{\lambda_1(t,x)}, \ldots, \xi_K\sqrt{\lambda_K(t,x)})$

$\overline{\sigma}(t,z) = (\xi_1\sqrt{\nu_1(t,z)}, \ldots, \xi_K\sqrt{\nu_K(t,z)})$

$\sqrt{N}(\overline{I}^N - \overline{I}^\infty)$



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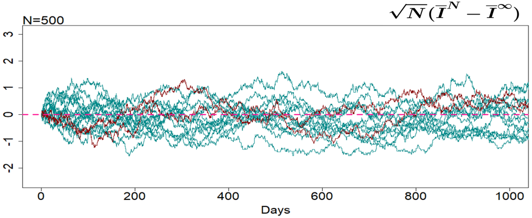

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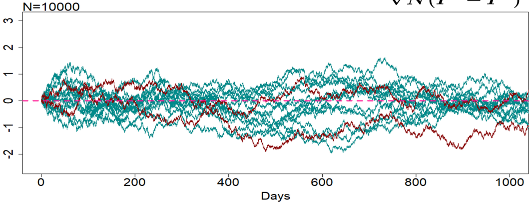
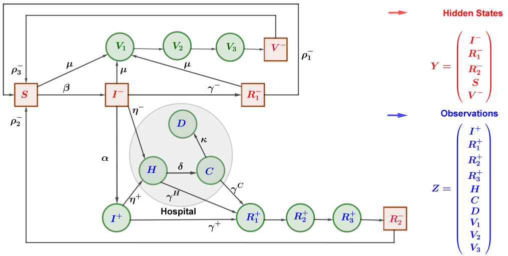
$\overline{\sigma}(t,z) = (\xi_1\sqrt{\nu_1(t,z)}, \dots, \xi_K\sqrt{\nu_K(t,z)})$

$\sqrt{N}(\overline{I}^N - \overline{I}^\infty)$



# Diffusion Approximation of COVID-19 Model




$$X = \begin{pmatrix} Y \\ Z \end{pmatrix}$$

$$\begin{aligned} dY &= \quad \bar{f}(t,Y,Z)dt + \bar{\sigma}(t,Y,Z)dW^1 + \bar{g}(t,Y,Z)dW^2 \quad \text{hidden state} \\ dZ &= \underbrace{[\bar{h}_0(t,Z) + \bar{h}_1(t,Z)Y]}_{\text{linear in } Y} dt + \bar{\ell}(t,Y,Z)dW^2 \quad \text{observation} \end{aligned}$$

Coefficients $\bar{f}$, $\bar{\sigma}$, $\bar{g}$, $\bar{\ell}$ are non-linear in the hidden state $Y$

Time discretization, $t_n = n\Delta t$, $n = 0, 1, \ldots$

$$\begin{aligned} Y_{n+1} &= Y_n + \quad f(n, Y_n, Z_n) + \sigma(n, Y_n, Z_n)\mathcal{E}^1_{n+1} + g(n, Y_n, Z_n)\mathcal{E}^2_{n+1} \\ Z_{n+1} &= Z_n + [h_0(n, Z_n) + h_1(n, Z_n)Y_n] + \ell(n, Y_n, Z_n)\mathcal{E}^2_{n+1} \end{aligned}$$

$(\mathcal{E}^1_n), (\mathcal{E}^2_n)$ independent sequences of i.i.d. $\mathcal{N}(0, \mathbb{1})$ random vectors

Given the observations of $Z_n$ we want to estimate hidden state $Y_n$

# Filtering Problem

Decompose state vector $X = (Y, Z)^\top$ into

Y: hidden (non-observable) state

Z: observation

Given observations $Z_k$ for $k = 0, \dots, n$ and

$\mathcal{F}_0^I$ initial information about distribution of $Y_0$

Mean-square optimal estimate of $Y_n$ given $\mathcal{F}_n^Z = \sigma\{Z_k, k = 0, \dots, n\} \vee \mathcal{F}_0^I$ is

### Conditional Mean

$$M_n = \mathbb{E}\left[Y_n | \mathcal{F}_n^Z\right]$$

Measure of estimation error

### Conditional Covariance

$$Q_n := \mathrm{Var}(Y_n | \mathcal{F}_n^Z) = \mathbb{E}[(Y_n - M_n)(Y_n - M_n)^\top | \mathcal{F}_n^Z]$$

Initial estimates

$$M_0 = m_0 = \mathbb{E}[Y_0 | \mathcal{F}_0^Z] \quad \text{and}$$

$$Q_0 = q_0 = \mathrm{Var}(Y_0 | \mathcal{F}_0^Z)$$

# Kalman Filter for Conditionally Gaussian Sequences

$$Y_{n+1} = [f_0 + f_1 Y_n] + \sigma \mathcal{E}^1_{n+1} + g\, \mathcal{E}^2_{n+1} \qquad \text{hidden state/signal}$$

$$Z_{n+1} = [h_0 + h_1 Y_n] + \ell\, \mathcal{E}^2_{n+1} \qquad \text{observation}$$

$(\mathcal{E}^1_n), (\mathcal{E}^2_n)$ independent sequences of i.i.d. $\mathcal{N}(0, \mathbb{1})$ random vectors

Coefficients $a = f_0, f_1, h_0, h_1, \sigma, g, \ell$ are of the form $a = a(n, \mathcal{Z}_n)$

may depend on time $n$

and also on whole observation path $\mathcal{Z}_n = (Z_k)_{k \le n}$ up to time $n$

**Theorem (Liptser & Shiryaev (2001), Theorem 13.4)**

*Under technical assumptions the conditional distribution of $Y_n$ given $\mathcal{F}^Z_n$ is $\mathcal{N}(M_n, Q_n)$ (Gaussian).*

*$M_n$ and $Q_n$ are defined by the following recursions driven by the observations*

$$M_{n+1} = [f_0 + f_1 M_n] + \left[g\ell^\top + f_1 Q_n h_1^\top\right] \left[\ell\ell^\top + h_1 Q_n h_1^\top\right]^+ \left[Z_{n+1} - \left(h_0 + h_1 M_n\right)\right]$$

$$Q_{n+1} = -\left[g\ell^\top + f_1 Q_n h_1^\top\right] \left[\ell\ell^\top + h_1 Q_n h_1^\top\right]^+ \left[g\ell^\top + f_1 Q_n h_1^\top\right]^\top + f_1 Q_n f_1^\top + \sigma\sigma^T$$

*with initial values $M_0 = m_0$, $Q_0 = q_0$ and $Y_0 \sim \mathcal{N}(m_0, q_0)$.*

*($[A]^+$ denotes the pseudoinverse of $A$)*

Note that all coefficients may depend on time $n$ and the observation path $\mathcal{Z}_n$.

# Extended Kalman Filter

$$\begin{aligned} Y_{n+1} &= Y_n + f(n, Y_n, Z_n) + \sigma(n, Y_n, Z_n)\mathcal{E}^1_{n+1} + g(n, Y_n, Z_n)\mathcal{E}^2_{n+1} \\ Z_{n+1} &= Z_n + [h_0(n, Z_n) + h_1(n, Z_n)Y_n] + \ell(n, Y_n, Z_n)\mathcal{E}^2_{n+1} \end{aligned}$$

Drift coefficient $f$ non-linear w.r.t. signal $Y$

Diffusion coefficients $\sigma, g, \ell$ may also depend on signal $Y$

Idea: Gelb (1974), Pardoux (1991), Bain, Crisan (2009)

1. Linearize drift coefficient $f$ by Taylor expansion around a "suitable" $\overline{Y}_n$
2. Substitute signal $Y$ by $\overline{Y}_n$ in diffusion coefficients $\sigma, g, \ell$

### Approximation by Conditional Gaussian Sequences

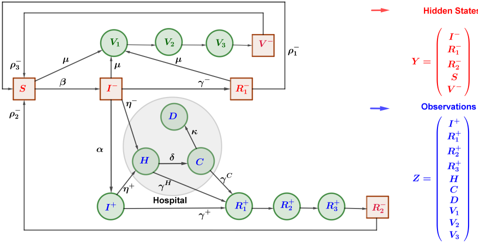
$$\begin{aligned} \widetilde{Y}_{n+1} &= \widetilde{Y}_n + f(n, \overline{Y}_n, \widetilde{Z}_n) + \frac{\partial f}{\partial y}(n, \overline{Y}_n, \widetilde{Z}_n)(\widetilde{Y}_n - \overline{Y}_n) \\ &\quad + \sigma(n, \overline{Y}_n, \widetilde{Z}_n)\mathcal{E}^1_{n+1} + g(n, \overline{Y}_n, \widetilde{Z}_n)\mathcal{E}^2_{n+1} \\ \widetilde{Z}_{n+1} &= \widetilde{Z}_n + [h_0(n, \widetilde{Z}_n) + h_1(n, \widetilde{Z}_n)\widetilde{Y}_n] + \ell(n, \overline{Y}_n, \widetilde{Z}_n)\mathcal{E}^2_{n+1} \end{aligned}$$

3. Apply Kalman filter for conditional Gaussian sequences $\rightsquigarrow (M_n, Q_n)$

Extended Kalman filter linearizes around current filter estimate: $\overline{Y}_n = M_n$.

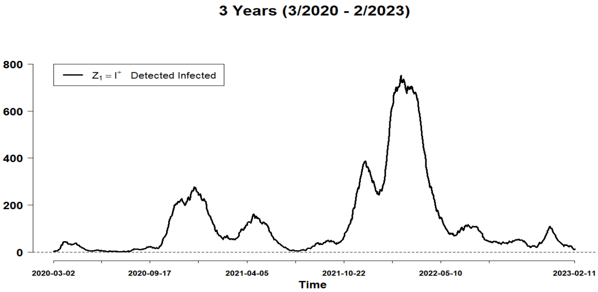
For theoretical justification and error estimates see Picard (1991)

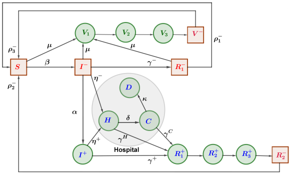
- COVID-19 model with partial information



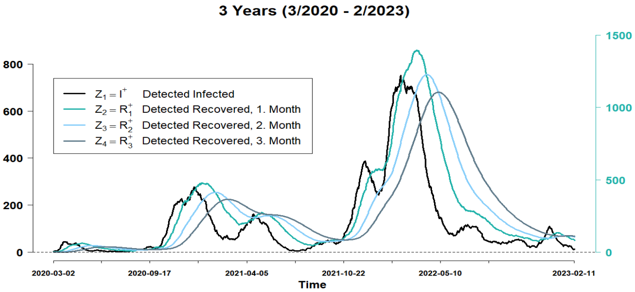

- Calibrate model parameters to real-world data set for Germany
  Time-depending $\beta$ and $\alpha$ match daily basic reproduction numbers,
  positive tests
- $T = 3$ years from March 2020 to February 2023, $\Delta t = 1$ day
- Population size $N = 100.000$
- Simulate hidden states and observations
- Compute filter estimates of hidden states based on observations
- Compare estimated and true values
  $\longrightarrow$ precision of the filter estimates

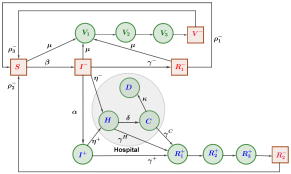
# Observations: Detected Infected



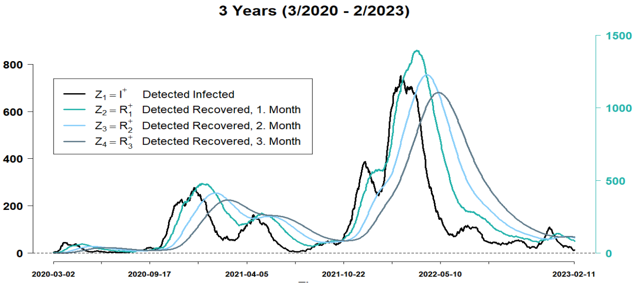



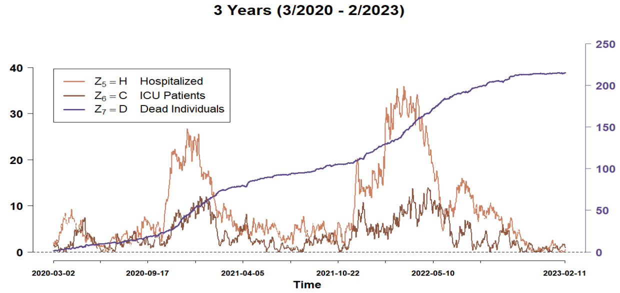
# Observations: Detected Infected & Recovered

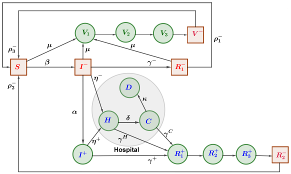


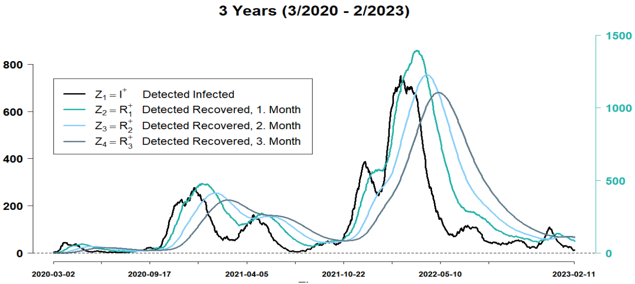
# Observations: Hospital

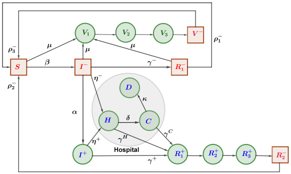


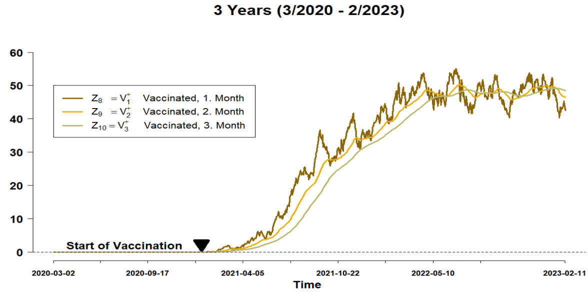



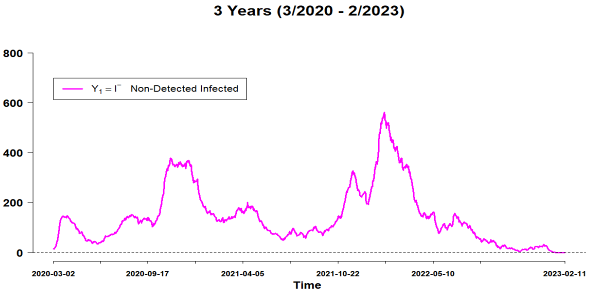
# Observations: Vaccinated

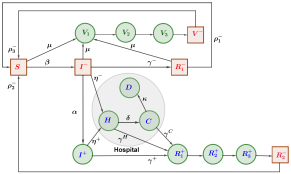


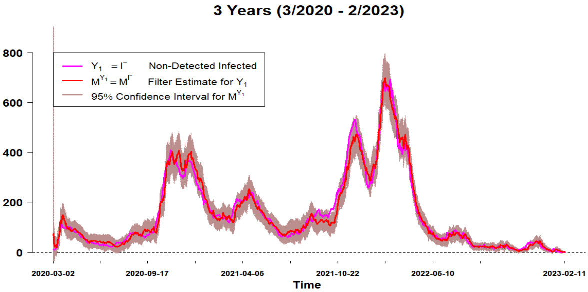



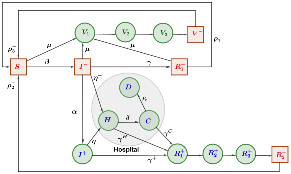
# Hidden Signal Simulation: Non-Detected Infected $I^-$



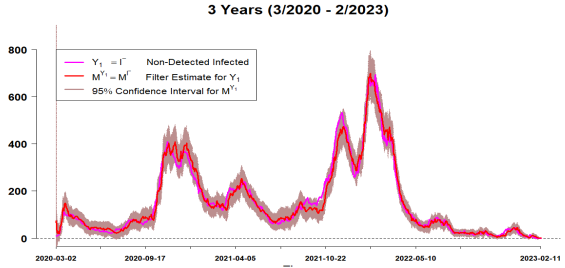



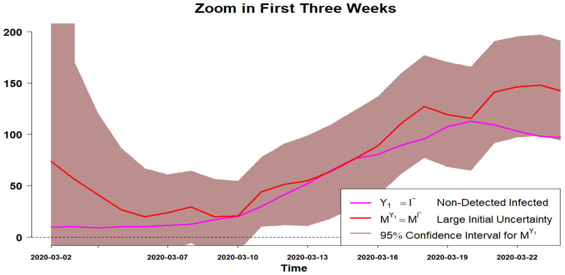
# Hidden Signal Estimation: Non-Detected Infected $I^-$

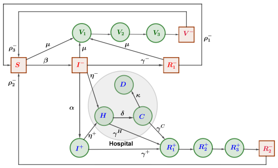


# Hidden Signal Estimation: Non-Detected Infected $I^-$

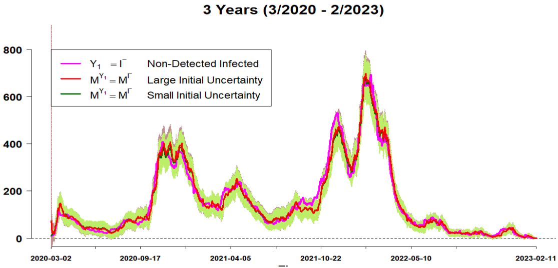


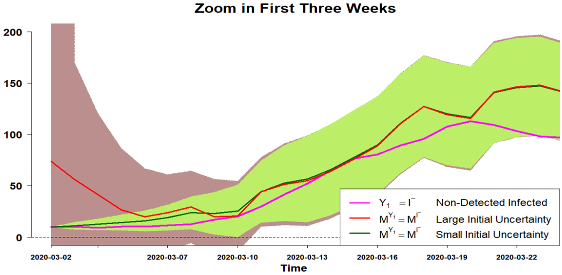



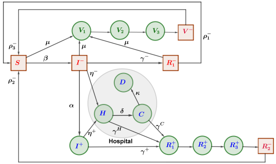




Large initial uncertainty is reduced by learning from observations

# Impact of Initial Estimate & Effect of Learning

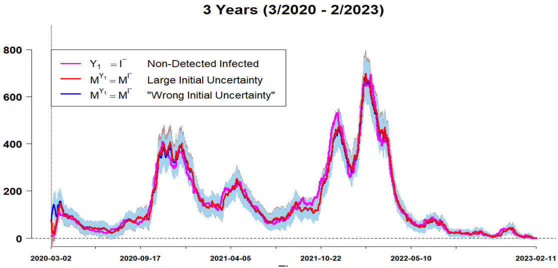


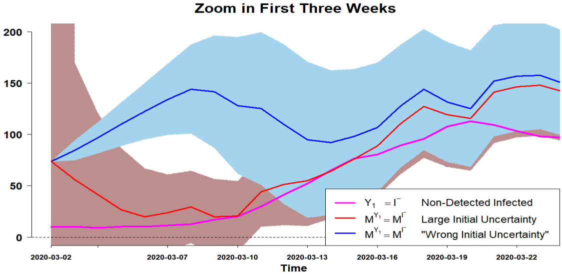



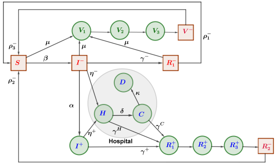




| | |
|---|---|
| Red | Large initial uncertainty is reduced by learning from observations |
| Green | Small initial uncertainty is fading out by observation noise |

# Impact of Initial Estimate & Effect of Learning




Red Large initial uncertainty is reduced by learning from observations

Blue "Wrong" initial uncertainty needs long time to be corrected

# Optimal Control Problem

For containment of an epidemics decision makers (government) try to influence the course of the epidemics by

1. Social distancing / lock-down with (relative) force $u_1 \in [0, 1]$, reduces transmission rate from $\beta$ to $(1 - u_1)\beta$
2. Tests/Diagnosis with intensity $u_2 \geq 0$
3. Vaccination with intensity $u_3 \geq 0$,

These measures have financial or social costs.

Available capacities for testing and vaccination are limited.

Aim: cost-optimal containment of the epidemics
through an appropriate mix of measures

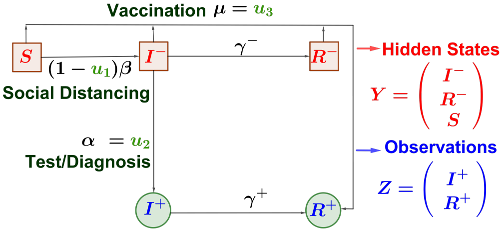
Decision-making problem under uncertainty about

future course of epidemics (forecasts),
current state of epidemics (nowcasts, dark numbers)

# Simplified Model

Model a disease with lifelong immunity after infection or vaccination

Example: measles



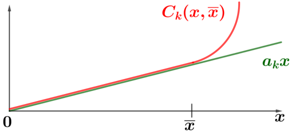
# Running and Terminal Cost

Running cost proportional to number $x$ of "affected" individuals,

Penalties if capacity threshold $\overline{x}$ is exceeded

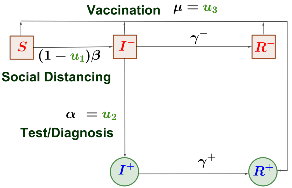
$C_k : \mathbb{R}^2 \to \mathbb{R}_+, \quad (x, \overline{x}) \mapsto C_k(x, \overline{x})$

increasing and convex functions w.r.t. $x$,




Example: $C_k(x, \overline{x}) = \begin{cases} a_k x, & x \le \overline{x} \\ a_k x + b_k(x - \overline{x})^2, & x > \overline{x} \end{cases}$

$\overline{x} = 0$: quadratic, $\overline{x} \to \infty$: linear




| | | |
|---|---|---|
| Social distancing (Lockdown) | $C_1(u_1 X^{\text{Work}}, 0)$, | $X^{\text{Work}} = N - I^+$ Charpentier et al. (2020) |
| Tests | $C_2(u_2 X^{\text{Test}}, \overline{x}^{\text{Test}})$, | $X^{\text{Test}} = N - I^+ - R^+$ |
| Vaccination | $C_3(u_3 X^{\text{Test}}, \overline{x}^{\text{Vacc}})$ | |
| Penalties | $C_4^{\pm}(I_{N_t}^{\pm}, 0)$ | for infected |

Running cost $\quad \Psi(X_n, u_n) = C_1 + C_2 + C_3 + C_4^+ + C_4^-$

Terminal cost $\quad \Phi(X_{N_t}) = C_T^+(I_{N_t}^+, 0) + C_T^-(I_{N_t}^-, 0)$

Penalties for infected at terminal time $T = N_t \Delta t$

# Performance Criterion

**Expected Aggregated Cost: Full Information, $X = (Y, Z)$**

$$\mathcal{J}^F(x;u) = \mathbb{E}\Big[\sum_{n=0}^{N_t-1} \Psi(X_n, u_n) + \Phi(X_{N_t})\Big| X_0 = x\Big]$$

Problem: $X = (Y, Z)$ depends on hidden state $Y$

Initial state $X_0 = x = (y, z)$ is not known

Decisions (control $u$) have to based on observable quantities $Z$ only

- Take conditional expectation w.r.t. $\mathcal{F}_0^Z$ (initial information)

$$\begin{aligned}\mathbb{E}[\mathcal{J}^F(X_0;u)|\mathcal{F}_0^Z] &= \mathbb{E}\Big[\sum_{n=0}^{N_t-1} \Psi(\underbrace{(Y_n, Z_n)}_{=X_n}, u_n) + \Phi(Y_{N_t}, Z_{N_t})\Big|\mathcal{F}_0^Z\Big] \\ &\overset{\substack{\text{tower law} \\ \text{Fubini}}}{=} \mathbb{E}\Big[\sum_{n=0}^{N_t-1} \mathbb{E}\Big[\Psi((Y_n, Z_n), u_n)\Big|\mathcal{F}_n^Z\Big] + \mathbb{E}\Big[\Phi(Y_{N_t}, Z_{N_t})\Big|\mathcal{F}_{N_t}^Z\Big]\Big|\mathcal{F}_0^Z\Big]\end{aligned}$$

- For $\Psi, \Phi$ linear and quadratic in $y$ conditional expectation $\mathbb{E}[\dots|\mathcal{F}_n^Z]$ can be expressed in terms of Extended Kalman filter $(M, Q)$ for hidden state $Y$.
  Recall: conditional distribution of $Y_n$ is Gaussian $\mathcal{N}(M_n, Q_n)$

**Performance Criterion: Partial Information, $X^P = (M, Q, Z)$**

$$\mathcal{J}(x^P;u) = \mathbb{E}\Big[\sum_{n=0}^{N_t-1} \Psi^P(X_n^P, u_n) + \Phi^P(X_{N_t}^P)\Big| X_0^P = x^P\Big]$$

# Optimal Control Problem with Partial Information

Replace hidden state $Y$ by filter $(M, Q)$

Rewrite dynamics of $M$ and $Z$ in terms of innovations process $(\overline{\mathcal{E}}_n)$

with $\quad \overline{\mathcal{E}}_n = ([\ell\ell^\top + h_1 Q_{n-1} h_1^\top]^+)^{1/2}(Z_n - (h_0 + h_1 M_{n-1})),$

$\quad (\overline{\mathcal{E}}_n)$ i.i.d. $\mathcal{N}(0, \mathbb{1})$ r.v.'s with $\mathcal{F}_n^Z = \mathcal{F}_n^{\overline{\mathcal{E}}} \vee \mathcal{F}_0^I$ (Liptser, Shiryaev (2001))

Treat control problem as Markov decision process (MDP) with

- state process $\quad X^P = (M, Q, Z)^\top$ taking values in state space $\mathcal{X}$
- dynamics $\quad X_{n+1}^P = \mathcal{T}(n, X_n^P, u_n, \overline{\mathcal{E}}_{n+1})$ with transition operator $\mathcal{T}$
- and Gaussian transition kernel.

$X^P = (M, Q, Z)$ is adapted to the observable filtration

$$\mathbb{F}^Z = (\mathcal{F}_n^Z)_{n \geq 0} \quad \text{with} \quad \mathcal{F}_n^Z = \sigma\{Z_k, k \leq n\} \vee \mathcal{F}_0^I$$

Admissible controls

$$\mathcal{A} = \{(u_n)_{n=0,\dots,N_t-1} \mid \mathbb{F}^Z\text{-adapted}, \ \text{integrability cond., Markov control } u_n = \widetilde{u}(n, X_n^P), \ \text{control constraints } u_n \in \mathcal{U} = [0,1] \times \mathbb{R}_+^2\}$$

Performance criterion for $n = 0, \dots, N_t$, $x = (m, q, z)^\top$ and $u \in \mathcal{A}$

$$J(n, x; u) = \mathbb{E}\Big[\sum_{k=n}^{N_t-1} \Psi^P(X_k^P, u_k) + \Phi^P(X_{N_t}^P) \Big| X_n^P = x\Big]$$

**Optimization problem**

Find $u^* \in \mathcal{A}$ such that $\quad J(n, x; u^*) = V(n, x) := \inf_{u \in \mathcal{A}} J(n, x; u)$

# Solution Using Dynamic Programming

**Bellman Equation / Dynamic Programming Equation**

$$V(n,x) = \inf_{\nu \in \mathcal{U}} \Big\{ \Psi^P(x,\nu) + \mathbb{E}_{n,x}\big[V(n+1, \underbrace{\mathcal{T}(n,x,\nu,\overline{\mathcal{E}}_{n+1})}_{=X^P_{n+1}})\big] \Big\}, \; n = 0,1,\dots,N-1$$

$$V(N_t, x) = \Phi^P(x) \qquad \text{(terminal condition)}$$

...can be solved by backward recursion

**Challenge**
- Compute $\mathbb{E}_{n,x}\big[V(n+1, \mathcal{T}(n,x,\nu,\overline{\mathcal{E}}_{n+1}))\big]$ at each time $n$ for all $x \in \mathcal{X}$!
- No closed-form expressions of the expectation are available.
- For high-dimensional state this becomes computationally intractable.
- $\rightarrow$ Curse of dimensionality
- Simplified model: dimension of state $X^P = (M, Q, Z)^\top$ is 7.

**Remedies**
- Apply quantization techniques as in Pagès (2015), Callegaro et al. (2017)
- Model order reduction: PCA for covariance matrix $Q$
- Q-Learning
- ⋮

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